Phys 321
Spring 2023

Quantum Theory I: Homework 18

Due: 11 April 2023

1 Gaussian wavepacket: momentum wavefunction

Consider a free particle in the state associated with wavefunction

$$\Psi(x) = \left(\frac{1}{\pi a^2}\right)^{1/4} e^{-(x-x_0)^2/2a^2}$$

where $a > 0$.

a) Show that this is normalized.

b) How would you describe the particle in terms of position?

c) Show that the associated momentum space wavefunction is

$$\tilde{\Psi}(p) = \left(\frac{a^2}{\pi \hbar^2}\right)^{1/4} e^{-ipx_0/\hbar}\, e^{-p^2 a^2/2\hbar^2}.$$

How would you describe the momentum of the particle in this state?

Now suppose that the particle is in the state

$$\Psi(x) = \left(\frac{1}{\pi a^2}\right)^{1/4} e^{-(x-x_0)^2/2a^2} e^{ip_0 x/\hbar}$$

where $a > 0$.

d) Show that the associated momentum space wavefunction is

$$\tilde{\Psi}(p) = \left(\frac{a^2}{\pi \hbar^2}\right)^{1/4} e^{-ipx_0/\hbar}\, e^{-(p-p_0)^2 a^2/2\hbar^2}.$$

How would you describe the particle in terms of its position and momentum if it were known to be in this state?

2 Free particle expectation value evolution: Gaussian wavepacket

A free particle particle is initially in the state corresponding to the wavefunction

$$\Psi(x,0) = \left(\frac{1}{\pi a^2}\right)^{1/4} e^{-x^2/2a^2}$$

where $a > 0$. This is normalized.

a) Determine the expectation values of position and momentum at *any later time*.
b) Determine the uncertainties in position and momentum at *any later time*.

3 Ehrenfest's theorem

In general Ehrenfest's theorem states that the expectation value of any time-independent observable satisfies:

$$\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle\Psi(t)|\left[\hat{H},\hat{A}\right]|\Psi(t)\rangle$$

where $\hat{H}$ is the Hamiltonian for the system. For particles in one dimension this can be used with the assistance of the following:

$$\begin{aligned}[\hat{x}^n,\hat{x}] &= 0\\ [\hat{p}^n,\hat{p}] &= 0\\ [\hat{x}^n,\hat{p}] &= i\hbar n\hat{x}^{n-1}\\ [\hat{p}^{\,n},\hat{x}] &= -i\hbar n\hat{p}^{\,n-1}\end{aligned}$$

These can be proved with fairly straightforward algebra.

a) Consider a free particle of mass m. Find expressions for

$$\frac{d\langle x\rangle}{dt} \quad\text{and}\quad \frac{d\langle p\rangle}{dt}$$

in terms of $\langle x\rangle$ and $\langle p\rangle$ and relevant constants. Use these to find a single differential equation for $\langle x\rangle$ which does not refer to $\langle p\rangle$. Compare this to the equation for the position of a free particle in classical mechanics.

b) Consider a harmonic oscillator for which

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2.$$

Find expressions for

$$\frac{d\langle x\rangle}{dt} \quad\text{and}\quad \frac{d\langle p\rangle}{dt}$$

in terms of $\langle x\rangle$ and $\langle p\rangle$ and relevant constants. Find a differential equation for $\langle x\rangle$ which does not refer to $\langle p\rangle$. Compare this to the equation for the position of a classical harmonic oscillator.

Ehrenfest's theorem provides one link between quantum mechanics and classical mechanics.