Quantum Theory I: Homework 13

Due: 17 March 2023

1 States of general systems

The energy eigenstates of a generic quantum system are $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, \ldots$ and the associated energies are $E_1, E_2, E_3, \ldots$. Consider the special states:

$$|\Psi\rangle := A\big[2i\,|\phi_1\rangle - |\phi_2\rangle + 5\,|\phi_3\rangle\big]$$
$$|\Phi\rangle := B\big[2\,|\phi_2\rangle - 5i\,|\phi_3\rangle + |\phi_4\rangle\big]$$

a) Determine the normalization constants A and B.

b) Determine $\langle\Phi|\Psi\rangle$.

c) Suppose that the energy of the system is measured for each state. List the outcomes and the probabilities with which they occur.

2 Generic measurement states

The energy eigenstates of a generic quantum system are $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, \ldots$. Consider the special states:

$$|\psi_1\rangle := \frac{1}{\sqrt{2}}\big[|\phi_1\rangle + |\phi_2\rangle\big]$$
$$|\psi_2\rangle := \frac{1}{2}\big[-|\phi_1\rangle + |\phi_2\rangle + \sqrt{2}\,|\phi_3\rangle\big]$$
$$|\psi_3\rangle := \frac{1}{2}\big[|\phi_1\rangle - |\phi_2\rangle + \sqrt{2}\,|\phi_3\rangle\big]$$

Do these states correspond to distinct outcomes of a single measurement? Explain your answer.

3 Coherent light states

A state of light has the form

$$|\Psi\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,.$$

where α is a constant and $|n\rangle$ represents a state with exactly n photons.

a) Suppose that the photon number is measured. List the outcomes and their probabilities.

b) Determine an expression for the expectation value of the number of photons.

c) Suppose $\alpha = 2$ that the photon number is measured. Determine an expression for the probabilities for $n = 0, 1, \ldots, 10$. Plot the probabilities versus n. How does the most likely value of n (that which gives the largest probability) compare to $\langle n \rangle$?

4 Particle states and measurements

For a particle of mass m in an infinite well of width L the energy eigenvalues are

$$E_n = \frac{\pi^2 \hbar^2 n^2}{2mL^2}$$

where $n = 1, 2, 3, \ldots$. Denote the associated state by $|\phi_n\rangle$. Suppose that a particle is in the following superposition of energy eigenstates at $t = 0$:

$$|\Psi(0)\rangle = A \sum_{n=1}^{\infty} \frac{1}{n^2} \, |\phi_n\rangle$$

where A is a real constant.

a) Apply the normalization condition to determine A.

b) Determine an expression for the expectation value of energy at time $t = 0$.

c) Determine an expression for the state at a later time t and use this to determine an expression for the expectation value of energy at time t. Verify that this does not depend on t.

Hint: the following will be essential:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$
$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$