Quantum Theory I: Homework 3

Due: 3 February 2023

1 Ket operations

Consider the kets

$$|\phi\rangle = \frac{1}{\sqrt{13}}\left(2\,|+\hat{\mathbf{z}}\rangle + 3\,|-\hat{\mathbf{z}}\rangle\right) \text{ and}$$
$$|\psi\rangle = \frac{1}{\sqrt{13}}\left(2\,|+\hat{\mathbf{z}}\rangle + 3i\,|-\hat{\mathbf{z}}\rangle\right)$$

Let

$$|\chi\rangle := \frac{1}{\sqrt{2}}\,|\phi\rangle + \frac{1}{\sqrt{2}}\,|\psi\rangle\,.$$

a) Express $|\chi\rangle$ as a combination of $|+\hat{\mathbf{z}}\rangle$ and $|-\hat{\mathbf{z}}\rangle$.
b) Show that each of $|\phi\rangle$ and $|\psi\rangle$ is normalized.
c) Check whether $|\chi\rangle$ is normalized.

2 Ket orthonormality

Consider the following kets:

$$|\phi\rangle = 12\,|+\hat{\mathbf{z}}\rangle - 5\,|-\hat{\mathbf{z}}\rangle \text{ and}$$
$$|\psi\rangle = 3i\,|+\hat{\mathbf{z}}\rangle + 4\,|-\hat{\mathbf{z}}\rangle\,.$$

a) Determine whether each ket is normalized. If not, describe how to normalize it, e.g. find A so that $A\,|\phi\rangle$ is normalized.
b) Determine the inner products $\langle\phi|\psi\rangle$ and $\langle\psi|\phi\rangle$.
c) For each of these kets, find another non-zero ket that is orthogonal to it.

3 Basis kets

Consider the pair of kets

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}\,|+\hat{\mathbf{z}}\rangle + \frac{1}{\sqrt{2}}\,|-\hat{\mathbf{z}}\rangle\,,$$
$$|\phi_2\rangle = \frac{1}{\sqrt{2}}\,|+\hat{\mathbf{z}}\rangle - \frac{1}{\sqrt{2}}\,|-\hat{\mathbf{z}}\rangle$$

and also the pair of kets

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\,|+\hat{\mathbf{z}}\rangle + \frac{i}{\sqrt{2}}\,|-\hat{\mathbf{z}}\rangle\,,$$
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\,|+\hat{\mathbf{z}}\rangle - \frac{i}{\sqrt{2}}\,|-\hat{\mathbf{z}}\rangle\,.$$

a) Express $|+\hat{\mathbf{z}}\rangle$ and $|-\hat{\mathbf{z}}\rangle$ as linear combinations of $|\phi_1\rangle$ and $|\phi_2\rangle$. That is write each in the form:
$$|+\hat{\mathbf{z}}\rangle = \text{number}\, |\phi_1\rangle + \text{number}\, |\phi_2\rangle\, .$$

b) Express $|+\hat{\mathbf{z}}\rangle$ and $|-\hat{\mathbf{z}}\rangle$ as linear combinations of $|\psi_1\rangle$ and $|\psi_2\rangle$.

c) Show that each of $|\phi_1\rangle$ and $|\phi_2\rangle$ is normalized and show that these are orthogonal.

d) Show that each of $|\psi_1\rangle$ and $|\psi_2\rangle$ is normalized and show that these are orthogonal.

e) Determine $\langle\phi_i|\psi_j\rangle$ and $|\langle\phi_i|\psi_j\rangle|^2$ for all combinations of i and j.

f) Describe whether it is possible that $|\phi_1\rangle$ and $|\phi_2\rangle$ are associated with the two outcomes of *one* particular measurement.

g) Describe whether it is possible that $|\psi_1\rangle$ and $|\psi_2\rangle$ are associated with the two outcomes of *one* particular measurement.

h) Describe whether it is possible that $|\phi_1\rangle$ and $|\psi_2\rangle$ are associated with the two outcomes of *one* particular measurement.

i) Describe whether it is possible that $|\psi_1\rangle$ and $|\phi_2\rangle$ are associated with the two outcomes of *one* particular measurement.

4 Kets and measurements

A spin-1/2 particle is known to be in the state
$$|\psi\rangle = \frac{3}{5}\, |+\hat{\boldsymbol{z}}\rangle + e^{i\phi}\frac{4}{5}\, |-\hat{\boldsymbol{z}}\rangle$$
where ϕ is real.

This particle is subjected to a SG $\hat{\boldsymbol{z}}$ measurement. The task is to predict the probabilities with which the two measurement outcomes occur.

a) What do you have to verify about the state before using it to determine the probabilities of the measurement outcomes?

b) Determine the probabilities of each measurement outcome. Do these depend on ϕ?

The two states
$$|\phi_1\rangle = \frac{1}{\sqrt{2}}\, |+\hat{\mathbf{z}}\rangle + \frac{1}{\sqrt{2}}\, |-\hat{\mathbf{z}}\rangle\, ,$$
$$|\phi_2\rangle = \frac{1}{\sqrt{2}}\, |+\hat{\mathbf{z}}\rangle - \frac{1}{\sqrt{2}}\, |-\hat{\mathbf{z}}\rangle$$
are orthonormal and therefore associated with the two outcomes of one particular measurement. At this stage we don't know what the measurement is. However, call the outcome associated with the first "plus" and that with the second "minus."

c) Determine the probabilities of each measurement outcomes for this unknown measurement. Do these depend on ϕ?