Quantum Theory I: Class Exam II

21 April 2023

Name: Solution **Total:** /50

Instructions

- There are 7 questions on 12 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$
Planck's constant	$h = 6.63 \times 10^{-34}\,\mathrm{Js}$ $\quad\hbar = 1.05 \times 10^{-34}\,\mathrm{Js}$
Mass of electron	$m_e = 9.11 \times 10^{-31}\,\mathrm{kg} = 511 \times 10^3\,\mathrm{eV/c^2}$
Mass of proton	$m_p = 1.673 \times 10^{-27}\,\mathrm{kg} = 938.3 \times 10^6\,\mathrm{eV/c^2}$
Mass of neutron	$m_n = 1.675 \times 10^{-27}\,\mathrm{kg} = 939.6 \times 10^6\,\mathrm{eV/c^2}$
Spherical coordinates	$\hat{\mathbf{n}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$
Spin 1/2 state	$\|+\hat{n}\rangle = \cos(\theta/2)\|+\hat{z}\rangle + e^{i\phi}\sin(\theta/2)\|-\hat{z}\rangle$
Spin 1/2 state	$\|-\hat{n}\rangle = \sin(\theta/2)\|+\hat{z}\rangle - e^{i\phi}\cos(\theta/2)\|-\hat{z}\rangle$
Euler relation	$e^{i\alpha} = \cos\alpha + i\sin\alpha$
Spin observables	$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
Rep. in $\|\pm\hat{\mathbf{z}}\rangle$ basis	$\hat{R}(\varphi n) = \begin{pmatrix}\cos\left(\frac{\varphi}{2}\right) - i\sin\left(\frac{\varphi}{2}\right)\cos\theta & -i\sin\left(\frac{\varphi}{2}\right)e^{-i\phi}\sin\theta\\ -i\sin\left(\frac{\varphi}{2}\right)e^{i\phi}\sin\theta & cos\left(\frac{\varphi}{2}\right) + i\sin\left(\frac{\varphi}{2}\right)\cos\theta\end{pmatrix}$

Physical constants and useful formulae

$$\sum_{n=1}^{\infty} a^n = \frac{a}{1-a} \quad \text{if } |a| < 1.$$

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} = e^a$$

$$\sum_{n=0}^{\infty} n\frac{a^n}{n!} = ae^a$$

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int x\sin(ax)\,\mathrm{d}x = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a}$$

$$\int x^2\sin(ax)\,\mathrm{d}x = \frac{2x^2}{a}\sin(ax) + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos(ax)$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x}\,\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}\,e^{\beta^2/4\alpha}$$

$$\int_{-\infty}^{\infty} xe^{-\alpha x^2+\beta x}\,\mathrm{d}x = \frac{\beta\sqrt{\pi}}{2\alpha^{3/2}}\,e^{\beta^2/4\alpha}$$

$$\int_{-\infty}^{\infty} x^2e^{-\alpha x^2+\beta x}\,\mathrm{d}x = \frac{(\beta^2+2\alpha)\sqrt{\pi}}{4\alpha^{5/2}}\,e^{\beta^2/4\alpha}$$

$$\int_{-\infty}^{\infty} x^3e^{-\alpha x^2+\beta x}\,\mathrm{d}x = \frac{\beta(\beta^2+6\alpha)\sqrt{\pi}}{8\alpha^{7/2}}\,e^{\beta^2/4\alpha}$$

Question 1

The Hamiltonian for a spin-1/2 particle in a magnetic field is

$$\hat{H} = \frac{\hbar\omega}{2}\hat{\sigma}_y$$

where, in the $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$ basis,

$$\hat{\sigma}_y \leftrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

a) The particle is initially in the state $|+\hat{z}\rangle$ and is allowed to evolve for time t. At that instant, S_z is measured. Determine the probabilities with which the measurement yields each outcome. 8

$$|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}|+\hat{z}\rangle$$

Then

$$e^{-i\hat{H}t/\hbar} = e^{-i\omega t\hat{\sigma}_y/2}$$

$$= \sum_{n=0} \frac{1}{n!}\left(\frac{-i\omega t}{2}\right)^n \hat{\sigma}_y^n$$

$$\hat{\sigma}_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{I}$$

$$\hat{\sigma}_y^3 = \hat{\sigma}_y\hat{I}$$

$$\hat{\sigma}_y^4 = \hat{\sigma}_y^2\hat{I} = \hat{I} \quad \text{etc..}$$

$$e^{-i\hat{H}t/\hbar} = \left[1 - \frac{1}{2!}\left(\frac{\omega t}{2}\right)^2 + ..\right]\hat{I} - i\left[\left(\frac{\omega t}{2}\right) - \frac{1}{3!}\left(\frac{\omega t}{2}\right)^3 + ..\right]\hat{\sigma}_y$$

$$= \cos\left(\frac{\omega t}{2}\right)\hat{I} - i\sin\left(\frac{\omega t}{2}\right)\hat{\sigma}_y$$

$$= \begin{pmatrix} \cos\frac{\omega t}{2} & -\sin\frac{\omega t}{2} \\ \sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} \end{pmatrix}$$

Question 1 continued ...

Thus

$$|\Psi(t)\rangle = \begin{pmatrix} \cos\frac{\omega t}{2} & -\sin\frac{\omega t}{2} \\ \sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\omega t}{2}\right) \\ \sin\left(\frac{\omega t}{2}\right) \end{pmatrix}$$

$$|\Psi(t)\rangle = \cos\left(\frac{\omega t}{2}\right) |+\hat{z}\rangle + \sin\left(\frac{\omega t}{2}\right) |-\hat{z}\rangle$$

Then

$$\boxed{\begin{aligned} \text{Prob}\,(S_z = +\hbar/2) &= |\langle +\hat{z}|\Psi(t)\rangle|^2 \\ &= \cos^2\left(\frac{\omega t}{2}\right) \\ \text{Prob}\,(S_z = -\hbar/2) &= |\langle -\hat{z}|\Psi(t)\rangle|^2 \\ &= \sin^2\left(\frac{\omega t}{2}\right) \end{aligned}}$$

b) Suppose that rather than measure S_z the component of spin along some other axis is measured. Describe whether there exists an axis direction such that the probabilities of the measurement outcomes do not depend on time. Explain your answer. 2

Yes. The evolution is a rotation about $\hat{y}$. This will leave the y-component of spin unaltered. Thus measure S_y - along the y-direction.

Note: $\text{Prob}\,(S_y = +\hbar/2) = |\langle +\hat{y}|\Psi(t)\rangle|^2$

$$\langle +\hat{y}| = \frac{1}{\sqrt{2}}\langle +\hat{z}| - \frac{i}{\sqrt{2}}\langle -\hat{z}| \Rightarrow \langle +\hat{y}|\Psi(t)\rangle = \frac{1}{\sqrt{2}}\left(\cos\frac{\omega t}{2} - i\sin\frac{\omega t}{2}\right)$$

$$= \frac{1}{\sqrt{2}} e^{-i\omega t/2}$$

$\Rightarrow \text{Prob}\,(S_y = +\hbar/2) = \frac{1}{2}$ independent of time. /10

Question 2

A spin-1/2 particle is in a region with uniform constant magnetic field. The state of the particle evolves according to

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}\,|+\hat{z}\rangle - \frac{1}{\sqrt{2}}\,e^{-i\omega t}\,|-\hat{z}\rangle\,.$$

Which of the following represents the magnetic field that causes this evolution?

i) $\mathbf{B} = B_0\hat{\mathbf{x}}$
ii) $\mathbf{B} = B_0\hat{\mathbf{y}}$
iii) $\mathbf{B} = B_0\hat{\mathbf{z}}$
iv) $\mathbf{B} = B_0\left[\frac{1}{\sqrt{2}}\hat{\mathbf{x}} + \frac{1}{\sqrt{2}}\hat{\mathbf{z}}\right]$
v) $\mathbf{B} = B_0\left[\frac{1}{\sqrt{2}}\hat{\mathbf{x}} - \frac{1}{\sqrt{2}}\hat{\mathbf{z}}\right]$
vi) $\mathbf{B} = B_0\left[\frac{1}{\sqrt{2}}\hat{\mathbf{z}} - \frac{1}{\sqrt{2}}\hat{\mathbf{z}}\right]$

Briefly explain your answer.

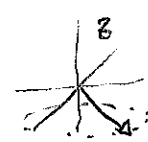
Any constant magnetic field will cause the state to rotate about an axis along the field direction, in the sense

$$|\Psi\rangle = |\hat{n}(t)\rangle$$

where $\hat{n}(t)$ is a rotating unit vector. In this case, for

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{1}{\sqrt{2}}e^{-i\omega t}|-\hat{z}\rangle$$

the angular co-ordinates for $\hat{n}$ are $\theta = \frac{3\pi}{2}$ $\phi = -\omega t$. So this is a rotation about $\hat{z}$ through angle ωt.



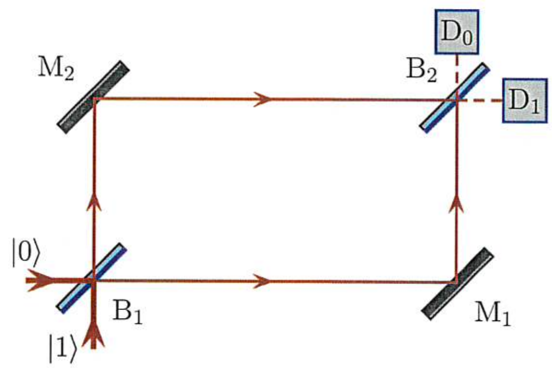
$\Rightarrow$ $\vec{B}$ is along $\hat{z}$

/5

Question 3

Do **either** part a) or part b) for full credit.

a) A Mach-Zehnder interferometer consists of an arrangement of two beam splitters, B_1 and B_2, two mirrors, M_1 and M_2, and two detectors as illustrated. Note that the reflective side of B_1 is down and right and that of B_2 is up and left. Ignore the thickness of the glass in the beam-splitters. Each beamsplitter reflects and transmits with different probabilities. The unitary operator for the beam splitters are



$$\hat{U}_{B_1} \leftrightarrow \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \qquad \text{and} \qquad \hat{U}_{B_2} \leftrightarrow \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}.$$

Suppose that a single photon is in the state $|0\rangle$ prior to the first beam splitter. Determine the probability with which it will emerge in the detector D_0. Describe how this probability would change if B_2 were removed.

After B_1 the state is

$$|\Psi_1\rangle = U_{B_1}|0\rangle \leftrightarrow \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

After B_2 the state is

$$|\Psi_2\rangle = U_{B_2}|\Psi_1\rangle = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 9+16 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

Thus with B_2 present $\quad \text{Prob}(D_0) = |\langle 0|\Psi_2\rangle|^2 = 1$

$\text{Prob}(D_1) = |\langle 1|\Psi_2\rangle|^2 = 0$

with B_2 absent $\quad \text{Prob}(D_0) = |\langle 0|\Psi_1\rangle|^2 = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$

$\text{Prob}(D_1) = |\langle 1|\Psi_1\rangle|^2 = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$

Question 3 continued ...

b) A particle that moves in one dimension is in one of the following states,

$$|\Psi\rangle \leftrightarrow \Psi(x) = Ae^{-x^2/2a^2}$$
$$|\Phi\rangle \leftrightarrow \Phi(x) = Bx^2e^{-x^2/2a^2}$$

where A and B are constants. Determine an expression for the inner product $\langle\Phi|\Psi\rangle$. A Martian asserts that these two states are the states associated with distinct outcomes for one type of measurement. Describe whether this assertion is true or false.

$$\langle\Phi|\Psi\rangle = \int_{-\infty}^{\infty} \Phi^*(x)\Psi(x)dx$$

$$= AB\int_{-\infty}^{\infty} x^2 e^{-x^2/2a^2} e^{-x^2/2a^2} dx$$

$$= AB\int_{-\infty}^{\infty} x^2 e^{-x^2/a^2} dx$$

$$= \frac{2\,\frac{1}{a^2}\sqrt{\pi}}{4\,(a^{-2})^{5/2}} = \frac{a^3\sqrt{\pi}}{2}$$

The states cannot be associated with distinct outcomes of one measurement since they would need to be orthogonal for this to be true. They are not orthogonal.

/6

Question 4

Particles with mass m are in an infinite well with potential

$$V(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant L \\ \infty & \text{otherwise.} \end{cases}$$

The position space wavefunction corresponding to the energy eigenstate with energy $E_n = n^2\pi^2\hbar^2/2mL^2$ is

$$\phi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & 0 \leqslant x \leqslant L \\ 0 & \text{otherwise.} \end{cases}$$

It is known that every particle is in the same state, which can be expressed as

$$|\Psi\rangle = \sum_n c_n |\phi_n\rangle$$

where c_n are constants. An ensemble of 1000 such particles is used and the energy of each is measured. It is found that the outcome E_1 is attained 360 times and E_3 is attained 640 times. Determine possible values for the constants c_n, using the information about the outcomes of energy measurements.

$$\text{Prob}(E_1) = \frac{360}{1000} = \frac{9}{25} = |\langle\phi_1|\Psi\rangle|^2$$

$$= |c_1|^2 \Rightarrow c_1 = \frac{3}{5}$$

$$\text{Prob}(E_3) = \frac{640}{1000} = \frac{16}{25} = |\langle\phi_3|\Psi\rangle|^2$$

$$= |c_3|^2 \Rightarrow c_3 = \frac{4}{5}$$

All others are zero:

$$|\Psi\rangle = \frac{3}{5}|\phi_1\rangle + \frac{4}{5}|\phi_3\rangle$$

/6

Question 5

An ensemble of particles with mass m are in an infinite well with potential

$$V(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant L \\ \infty & \text{otherwise.} \end{cases}$$

At time $t = 0$, each particle is in the state

$$|\Psi(0)\rangle = \frac{3}{5}|\phi_2\rangle + \frac{4}{5}|\phi_4\rangle$$

where $|\phi_n\rangle$ is the energy eigenstate with eigenvalue $E_n = n^2\pi^2\hbar^2/2mL^2$.

Show that the expectation value of position measurements oscillates as time passes. Determine an expression for the frequency of oscillation in terms of m, L and $\hbar$.

$$\langle x \rangle = \langle \Psi(t)|\hat{x}|\Psi(t)\rangle$$

Then $$|\Psi(t)\rangle = \frac{3}{5}e^{-iE_2t/\hbar}|\phi_2\rangle + \frac{4}{5}e^{-iE_4t/\hbar}|\phi_4\rangle$$ +2

$$\Rightarrow \langle x \rangle = \left[\frac{3}{5}e^{iE_2t/\hbar}\langle\phi_2| + \frac{4}{5}e^{iE_4t/\hbar}\langle\phi_4|\right]\hat{x}\left[\frac{3}{5}e^{-iE_2t/\hbar}|\phi_2\rangle + \frac{4}{5}e^{-iE_4t/\hbar}|\phi_4\rangle\right]$$

$$= \frac{9}{25}\langle\phi_2|\hat{x}|\phi_2\rangle + \frac{16}{25}\langle\phi_4|\hat{x}|\phi_4\rangle$$

$$+ \frac{12}{25}\left[e^{i(E_4-E_2)t/\hbar}\langle\phi_4|\hat{x}|\phi_2\rangle + e^{-i(E_4-E_2)t/\hbar}\langle\phi_2|\hat{x}|\phi_4\rangle\right]$$

Then $$\langle\phi_2|\hat{x}|\phi_2\rangle = \int_0^L x|\phi_2(x)|^2 dx = \frac{2}{L}\int_0^L x\sin^2\left(\frac{2\pi x}{L}\right)dx = \frac{L}{2}$$

Similarly $\langle\phi_4|\hat{x}|\phi_4\rangle = \frac{L}{2}$. The other two are equal. Thus

(+4)

$$\langle x \rangle = \frac{L}{2} + \frac{12}{25}\langle\phi_4|\hat{x}|\phi_2\rangle\, 2\cos\left[\frac{(E_4-E_2)t}{\hbar}\right].$$ +2

The frequency of oscillation is $\omega = \frac{E_4-E_2}{\hbar}$

/8

$$= \frac{\pi^2\hbar}{2mL^2}(16-4) = \frac{6\pi^2\hbar}{mL^2}$$

Question 6

Consider the two position space wavefunctions

$$\Psi_1(x) = \begin{cases} \dfrac{1}{\sqrt{L}} e^{ip_0x/\hbar} & \text{if } -L/2 \leqslant x \leqslant L/2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Psi_2(x) = \begin{cases} \dfrac{1}{\sqrt{L}} e^{-ip_0x/\hbar} & \text{if } -L/2 \leqslant x \leqslant L/2 \\ 0 & \text{otherwise} \end{cases}$$

where $p_0 > 0$. *Note: they do differ by the sign in front of* p_0.

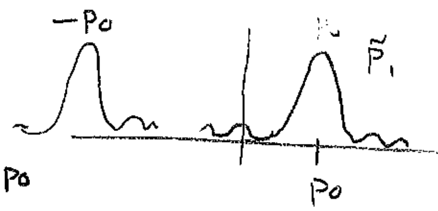
a) Determine the momentum space wavefunction and probability density for each.

$$\tilde{\Psi}_1(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-L/2}^{L/2} e^{ip_0x/\hbar} e^{-ipx/\hbar}\, dx = \frac{1}{\sqrt{2\pi\hbar}} \int_{-L/2}^{L/2} e^{i(p_0-p)x/\hbar}\, dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{\hbar}{i(p_0-p)} e^{i(p_0-p)x/\hbar}\Big|_{-L/2}^{L/2}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{\hbar}{i(p_0-p)} \underbrace{\left[e^{i(p_0-p)L/2\hbar} - e^{i(p_0-p)L/2\hbar}\right]}_{2i\sin(p_0-p)L/2\hbar} = \sqrt{\frac{2\hbar}{\pi}} \frac{\sin\left[\frac{(p_0-p)L}{2\hbar}\right]}{p_0-p}$$

$$\tilde{P}_1(p) = |\tilde{\Psi}_1(p)|^2 = \frac{2\hbar}{\pi} \frac{\sin^2\left[\frac{p_0-p}{2\hbar} L\right]}{(p_0-p)^2}$$



For Ψ_2 these are the same except $p_0 \to -p_0$

$$\tilde{P}_2(p) = |\tilde{\Psi}_2(p)|^2 = \frac{2\hbar}{\pi} \frac{\sin^2\left[\frac{p_0+p}{2\hbar} L\right]}{(p_0+p)^2}$$

b) Based on the possible outcomes of momentum measurements for particles in each state, how would you you describe each state in terms of momentum?

Ψ_1 corresponds to a particle with positive momentum in the vicinity of p_0

Ψ_2 " " " " " negative " " " " " $-p_0$

/8

Question 7

Do **either** part a) or part b) for full credit.

a) Consider a harmonic oscillator with mass m and angular frequency ω. The position wavefunction for the grouns state $|0\rangle$ is

$$\phi_0(x) = \left(\frac{m\omega}{4\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$$

Determine the position wavefunction for the energy eigenstate $|1\rangle$.

Consider $\hat{a}^\dagger |0\rangle = \sqrt{1}\, |1\rangle = |1\rangle$

$\Rightarrow \quad |1\rangle = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)|0\rangle$

$\Rightarrow \quad \phi_1(x) = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{i}{m\omega}\left(-i\hbar\frac{\partial}{\partial x}\right)\right)\phi_0(x)$

$= \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\phi_0(x)$

$\frac{\partial}{\partial x}\phi_0(x) = \left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left(-\frac{m\omega}{2\hbar}2x\right)e^{-m\omega x^2/2\hbar}$

$= -\left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\frac{m\omega}{\hbar}\, x\, e^{-m\omega x^2/2\hbar} = -\frac{m\omega}{\hbar}x\,\phi_0(x)$

$\Rightarrow \quad \phi_1(x) = \sqrt{\frac{m\omega}{2\hbar}}\left(x\phi_0(x) + x\,\phi_0(x)\right)$

$= 2\sqrt{\frac{m\omega}{2\hbar}}\ \phi_0(x)$

$= 2\sqrt{\frac{m\omega}{2\hbar}}\left(\frac{m\omega}{4\pi\hbar}\right)^{1/4} x\, e^{-m\omega x^2/2\hbar}$

Question 7 continued ...

b) Consider a harmonic oscillator with mass m and angular frequency ω. The oscillator is in the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\,|2\rangle + \frac{i}{\sqrt{2}}\,|3\rangle\,.$$

Determine expressions for the expectation values of **position** and **momentum** measurements.

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}+\hat{a}^+)$$

$$\langle x\rangle = \langle\Psi|\,\hat{x}\,|\Psi\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left[\langle 2|-i\langle 3|\right]\underbrace{\left[\hat{a}+\hat{a}^+\right]\left[|2\rangle+i|3\rangle\right]}$$

$$= \hat{a}|2\rangle + i\hat{a}|3\rangle + \hat{a}^+|2\rangle + i\hat{a}^+|3\rangle$$

$$= \sqrt{2}\,|1\rangle + i\sqrt{3}\,|2\rangle + \sqrt{3}\,|3\rangle + i\sqrt{4}\,|4\rangle$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left[i\sqrt{3}-i\sqrt{3}\right] = 0 \qquad \Rightarrow \langle x\rangle = 0$$

$$\langle p\rangle = -i\sqrt{\frac{\hbar m\omega}{2}}\,\langle\Psi|(\hat{a}-\hat{a}^+)|\Psi\rangle$$

$$= -i\sqrt{\frac{\hbar m\omega}{2}}\,\frac{1}{2}\left\{\langle 2|-i\langle 3|\right]\left[\sqrt{2}|1\rangle + i\sqrt{3}|2\rangle - \sqrt{3}|3\rangle - i\sqrt{4}\,|4\rangle\right]$$

$$= -i\,\frac{1}{2}\sqrt{\frac{\hbar\omega m}{2}}\;2i\sqrt{3}$$

$$\langle p\rangle = \sqrt{\frac{3\hbar m\omega}{2}}$$

/7