Quantum Theory I: Class Exam II

21 April 2022

Name: ____________________________ **Total:** **/50**

Instructions

- There are 5 questions on 10 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$
Planck's constant	$h = 6.63 \times 10^{-34}\,\mathrm{Js}$ $\qquad \hbar = 1.05 \times 10^{-34}\,\mathrm{Js}$
Mass of electron	$m_e = 9.11 \times 10^{-31}\,\mathrm{kg} = 511 \times 10^3\,\mathrm{eV/c^2}$
Mass of proton	$m_p = 1.673 \times 10^{-27}\,\mathrm{kg} = 938.3 \times 10^6\,\mathrm{eV/c^2}$
Mass of neutron	$m_n = 1.675 \times 10^{-27}\,\mathrm{kg} = 939.6 \times 10^6\,\mathrm{eV/c^2}$
Spherical coordinates	$\hat{\mathbf{n}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$
Spin 1/2 state	$\lvert +\hat{\boldsymbol{n}}\rangle = \cos(\theta/2)\lvert +\hat{\boldsymbol{z}}\rangle + e^{i\phi}\sin(\theta/2)\lvert -\hat{\boldsymbol{z}}\rangle$
Spin 1/2 state	$\lvert -\hat{\boldsymbol{n}}\rangle = \sin(\theta/2)\lvert +\hat{\boldsymbol{z}}\rangle - e^{i\phi}\cos(\theta/2)\lvert -\hat{\boldsymbol{z}}\rangle$
Euler relation	$e^{i\alpha} = \cos\alpha + i\sin\alpha$
Spin observables	$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Rep. in $\lvert\pm\hat{\mathbf{z}}\rangle$ basis	$\hat{R}(\varphi\boldsymbol{n}) = \begin{pmatrix} \cos\left(\frac{\varphi}{2}\right) - i\sin\left(\frac{\varphi}{2}\right)\cos\theta & -i\sin\left(\frac{\varphi}{2}\right)e^{-i\phi}\sin\theta \\ -i\sin\left(\frac{\varphi}{2}\right)e^{i\phi}\sin\theta & cos\left(\frac{\varphi}{2}\right) + i\sin\left(\frac{\varphi}{2}\right)\cos\theta \end{pmatrix}$

Physical constants and useful formulae

$$\sum_{n=1}^{\infty} a^n = \frac{a}{1-a} \quad \text{if } |a| < 1.$$

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} = e^a$$

$$\sum_{n=0}^{\infty} n\frac{a^n}{n!} = ae^a$$

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int x\sin(ax)\,\mathrm{d}x = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a}$$

$$\int x^2\sin(ax)\,\mathrm{d}x = \frac{2x^2}{a}\sin(ax) + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos(ax)$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x}\,\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}\,e^{\beta^2/4\alpha}$$

$$\int_{-\infty}^{\infty} xe^{-\alpha x^2+\beta x}\,\mathrm{d}x = \frac{\beta\sqrt{\pi}}{2\alpha^{3/2}}\,e^{\beta^2/4\alpha}$$

$$\int_{-\infty}^{\infty} x^2e^{-\alpha x^2+\beta x}\,\mathrm{d}x = \frac{(\beta^2+2\alpha)\sqrt{\pi}}{4\alpha^{5/2}}\,e^{\beta^2/4\alpha}$$

$$\int_{-\infty}^{\infty} x^3e^{-\alpha x^2+\beta x}\,\mathrm{d}x = \frac{\beta(\beta^2+6\alpha)\sqrt{\pi}}{8\alpha^{7/2}}\,e^{\beta^2/4\alpha}$$

Question 1

A spin-1/2 particle is initially in the state $|+\hat{\boldsymbol{y}}\rangle$ and it is then placed in a constant magnetic field $\mathbf{B} = B_0\hat{\mathbf{x}}$.

a) Show that the Hamiltonian has the form

$$\hat{H} = \frac{\hbar\omega}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

and determine an expression for ω in terms of the particle mass m, charge q and g-factor g.

b) Determine an expression for the state of the particle after it has been in the field for time t.

Question 1 continued ...

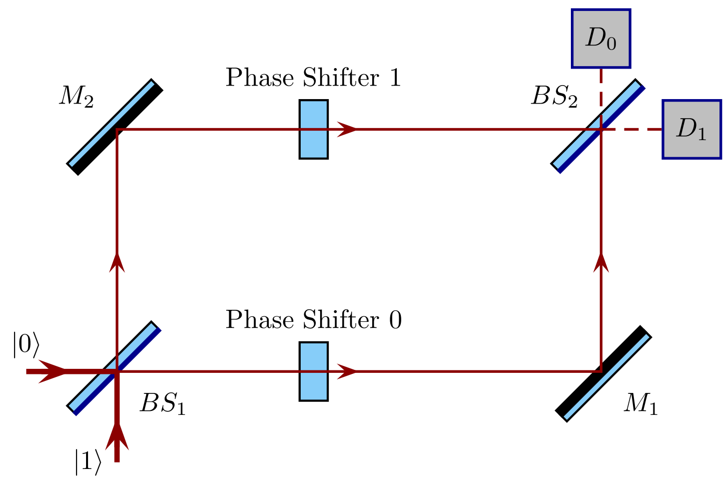
c) Determine an expression, in terms of ω_0, for the time should pass after the particle entered the field until it first reaches a state where an SG $\hat{\boldsymbol{z}}$ measurement yields $S_z = \hbar/2$ with certainty.

/12

Question 2

Do **either** part a) or part b) for full credit.

a) A Mach-Zehnder interferometer consists of an arrangement of two beam splitters, BS_1 and BS_2, two mirrors, M_1 and M_2, and two detectors as illustrated. Note that the reflective side of B_1 is down and right and that of B_2 is up and left. Ignore the thickness of the glass in the beam-splitters and assume that they reflect 50% of the beam and transmit 50%.



The unitary operator for the beam splitters are

$$\hat{U}_{\text{BS1}} \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \hat{U}_{\text{BS2}} \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

That for passage between the beam splitters (this includes the phase shifters) is

$$\hat{U}_{\text{PS}} = \begin{pmatrix} e^{i\varphi_0} & 0 \\ 0 & e^{i\varphi_1} \end{pmatrix}.$$

Suppose that a single photon is in the state $|0\rangle$ prior to the first beam splitter. Determine the probability with which it will emerge in the detector D_0. Determine the most general relationship between φ_0 and φ_1 such that the photon is equally likely to arrive in either detector.

Question 2 continued . . .

Question 2 continued . . .

b) A state of light can be represented by

$$|\Psi\rangle = e^{-9/2} \sum_{n=0}^{\infty} \frac{3^n}{\sqrt{n!}} |n\rangle$$

where $|n\rangle$ is a state such that a measurement of the number of photons yields exactly n. Suppose that the number of photons is measured. Determine the probability with which the measurement yields exactly zero photons **and** the expectation value of the number of photons.

/10

Question 3

Particles of mass m are each in an infinite well with potential

$$V(x) = \begin{cases} 0 & 0 \leqslant x \leqslant L \\ \infty & \text{otherwise.} \end{cases}$$

The (position) wavefunction corresponding to the n^{th} energy eigenstate, $|\phi_n\rangle$, with energy $E_n = n^2\pi^2\hbar^2/2mL^2$ is

$$\phi_n(x) = \begin{cases} \sqrt{\dfrac{2}{L}} \sin\left(\dfrac{n\pi x}{L}\right) & 0 \leqslant x \leqslant L \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that, at one instant, the position space wavefunction for the state of the particle is **restricted to half of the well** and is

$$\Psi(x) = \begin{cases} \sqrt{\dfrac{960}{L^5}}\, x \left(x - \dfrac{L}{2}\right) & 0 \leqslant x \leqslant \dfrac{L}{2} \\ 0 & \text{otherwise.} \end{cases}$$

This is normalized. Determine the probability with which an energy measurement will yield the outcome $E = 4\pi^2\hbar^2/2mL^2$.

Question 3 continued ...

/12

Question 4

Suppose that, at one instant, a particle in an infinite square well is known to be in one of the states

$$|\Psi_A\rangle = \frac{1}{\sqrt{2}}\,|\phi_2\rangle + \frac{i}{\sqrt{2}}\,|\phi_3\rangle$$
$$|\Psi_B\rangle = \frac{1}{\sqrt{2}}\,|\phi_2\rangle - \frac{i}{\sqrt{2}}\,|\phi_3\rangle$$

where $|\phi_n\rangle$ are energy eigenstates. Would measuring the energy of the particle enable you to determine whether the particle was in state A or state B with certainty, with some likelihood of success or neither? Explain your answer.

/6

Question 5

A particle can move in one dimension. Its position space wavefunction is given by

$$\Psi(x) = \left(\frac{1}{\pi a^2}\right)^{1/4} e^{ip_0 x/\hbar}\, e^{-x^2/2a^2}$$

where p_0 has units of momentum.

a) Determine an expression for the momentum space wavefunction.

b) Describe qualitatively what you would expect for the outcomes of a momentum measurement on a particle in this state.

/10