Quantum Theory I: Final Exam

19 May 2022

Name: ____________________________ **Total:** **/50**

Instructions

- There are 7 questions on 14 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$
Planck's constant	$h = 6.63 \times 10^{-34}\,\mathrm{Js}$ $\quad \hbar = 1.05 \times 10^{-34}\,\mathrm{Js}$
Mass of electron	$m_e = 9.11 \times 10^{-31}\,\mathrm{kg} = 511 \times 10^3\,\mathrm{eV/c^2}$
Mass of proton	$m_p = 1.673 \times 10^{-27}\,\mathrm{kg} = 938.3 \times 10^6\,\mathrm{eV/c^2}$
Mass of neutron	$m_n = 1.675 \times 10^{-27}\,\mathrm{kg} = 939.6 \times 10^6\,\mathrm{eV/c^2}$
Spherical coordinates	$\hat{\mathbf{n}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$
Spin 1/2 state	$\lvert +\hat{\boldsymbol{n}}\rangle = \cos(\theta/2)\lvert +\hat{\boldsymbol{z}}\rangle + e^{i\phi}\sin(\theta/2)\lvert -\hat{\boldsymbol{z}}\rangle$
Spin 1/2 state	$\lvert -\hat{\boldsymbol{n}}\rangle = \sin(\theta/2)\lvert +\hat{\boldsymbol{z}}\rangle - e^{i\phi}\cos(\theta/2)\lvert -\hat{\boldsymbol{z}}\rangle$
Euler relation	$e^{i\alpha} = \cos\alpha + i\sin\alpha$
Spin observables	$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
Rep. in $\lvert\pm\hat{\mathbf{z}}\rangle$ basis	$\hat{R}(\varphi\boldsymbol{n}) = \begin{pmatrix}\cos\left(\frac{\varphi}{2}\right) - i\sin\left(\frac{\varphi}{2}\right)\cos\theta & -i\sin\left(\frac{\varphi}{2}\right)e^{-i\phi}\sin\theta\\ -i\sin\left(\frac{\varphi}{2}\right)e^{i\phi}\sin\theta & cos\left(\frac{\varphi}{2}\right) + i\sin\left(\frac{\varphi}{2}\right)\cos\theta\end{pmatrix}$

Physical constants and useful formulae

Angular Momentum

$$\hat{L}_x = i\hbar\left(\sin\phi\,\frac{\partial}{\partial\theta} + \frac{\cos\theta\cos\phi}{\sin\theta}\,\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}_y = i\hbar\left(-\cos\phi\,\frac{\partial}{\partial\theta} + \frac{\cos\theta\sin\phi}{\sin\theta}\,\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}_z = -i\hbar\,\frac{\partial}{\partial\phi}$$

$$\hat{L}_+\left|l,m\right\rangle = \hbar\sqrt{l(l+1)-m(m+1)}\left|l,m+1\right\rangle$$

$$\hat{L}_-\left|l,m\right\rangle = \hbar\sqrt{l(l+1)-m(m-1)}\left|l,m-1\right\rangle$$

$$\sum_{n=1}^{\infty} a^n = \frac{a}{1-a} \quad \text{if } |a|<1.$$

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} = e^a$$

$$\sum_{n=0}^{\infty} n\frac{a^n}{n!} = ae^a$$

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int x\sin(ax)\,\mathrm{d}x = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a}$$

$$\int x^2\sin(ax)\,\mathrm{d}x = \frac{2x^2}{a}\sin(ax) + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos(ax)$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

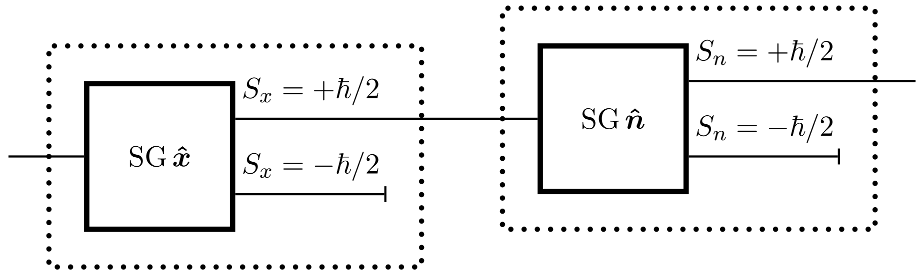
$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x}\,\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}\,e^{\beta^2/4\alpha} \qquad \int_{-\infty}^{\infty} xe^{-\alpha x^2+\beta x}\,\mathrm{d}x = \frac{\beta\sqrt{\pi}}{2\alpha^{3/2}}\,e^{\beta^2/4\alpha}$$

$$\int_{-\infty}^{\infty} x^2e^{-\alpha x^2+\beta x}\,\mathrm{d}x = \frac{(\beta^2+2\alpha)\sqrt{\pi}}{4\alpha^{5/2}}\,e^{\beta^2/4\alpha} \qquad \int_{-\infty}^{\infty} x^3e^{-\alpha x^2+\beta x}\,\mathrm{d}x = \frac{\beta(\beta^2+6\alpha)\sqrt{\pi}}{8\alpha^{7/2}}\,e^{\beta^2/4\alpha}$$

Question 1

Spin-1/2 particles are subjected to a sequence of Stern-Gerlach measurements. The first measures S_x and only particles that emerge with $S_x = +\hbar/2$ pass along to the second. The second can be oriented along various directions $\hat{\mathbf{n}}$. The $S_n = -\hbar/2$ output is blocked.



a) Suppose that $\hat{\mathbf{n}} = \hat{\mathbf{y}}$. Consider particles that emerge after the SG $\hat{\boldsymbol{x}}$ device. Determine the probability with which these emerge from the right dotted box.

b) Suppose that the entire illustrated sequence was followed with one more SG $\hat{\boldsymbol{x}}$ device. Considering all possible choices for $\hat{\mathbf{n}}$ describe whether or not the third measurement would yield $S_x = +\hbar/2$ *with certainty.*

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Question 2

An ensemble of spin-1/2 particles is each prepared in the state

$$|\Psi\rangle = \left[\frac{3}{5}\,|+\hat{\boldsymbol{z}}\rangle + \frac{4i}{5}\,|-\hat{\boldsymbol{z}}\rangle\right].$$

a) Determine the direction, $\hat{\mathbf{n}}$, along which an SG $\hat{\boldsymbol{n}}$ apparatus would have to be oriented so as to yield $S_n = +\hbar/2$ *with certainty.*

Question 2 continued . . .

b) Suppose that the entire ensemble were subjected to a measurement of S_y. Determine the expectation value $\langle S_y \rangle$ provided that every ensemble member were in the state $|\Psi\rangle$.

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Question 3

A spin-1/2 particle is placed in a uniform constant magnetic field that results in the Hamiltonian

$$\hat{H} = \frac{\hbar\omega}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

a) Suppose that the particle were initially ($t = 0$) in the state $|+\hat{\boldsymbol{z}}\rangle$. The particle subsequently evolves via the interaction with the magnetic field. Determine the state of the particle at any later time t and the probability that a measurement will yield $S_z = +\hbar/2$.

Question 3 continued ...

b) Describe the direction of the magnetic field so that it causes the evolution described above. Explain your answer.

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Question 4

The normalized energy eigenstates of some quantum system are denoted $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, \ldots$. Consider the special states:

$$
\begin{aligned}
|\psi_1\rangle &:= \frac{1}{\sqrt{2}}\Big[|\phi_1\rangle - |\phi_3\rangle\Big] \\
|\psi_2\rangle &:= \frac{1}{2}\Big[|\phi_1\rangle + \sqrt{2}\,|\phi_2\rangle + |\phi_3\rangle\Big] \\
|\psi_3\rangle &:= \frac{1}{2}\Big[|\phi_1\rangle - \sqrt{2}\,|\phi_2\rangle + |\phi_3\rangle\Big]
\end{aligned}
$$

Does these states correspond to distinct outcomes of a single measurement? Explain your answer.

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Question 5

A particle with mass m is in an infinite square well with potential,

$$V(x) = \begin{cases} 0 & 0 \leqslant x \leqslant L \\ \infty & \text{otherwise.} \end{cases}$$

The wavefunctions corresponding to normalized energy eigenstates, $|\phi_n\rangle$ for this system are

$$\phi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & 0 \leqslant x \leqslant L \\ 0 & \text{otherwise.} \end{cases}$$

At a particular instant the normalized state of the particle, $|\Psi\rangle$, corresponds to

$$\Psi(x) = \begin{cases} A & \frac{L}{4} \leqslant x \leqslant \frac{3L}{4} \\ 0 & \text{otherwise} \end{cases}$$

where $A > 0$ is a constant.

a) Determine the constant A.

Question 5 continued ...

b) Suppose that the energy of this particle is measured. List the possible outcomes and the probabilities with which they occur.

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Question 6

Do **either** part a) or part b) for full credit.

a) An ensemble of free particles with mass m are each in the state for which the normalized wavefunction is

$$\Psi(x) = \left(\frac{1}{\pi a^2}\right)^{1/4} e^{-x^2/2a^2}.$$

Determine the expectation value and the uncertainty of momentum measurements on this ensemble. Determine the expectation value of energy measurements on this ensemble.

Question 6 continued ...

b) An ensemble of quantum harmonic oscillators are each initially in the state

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\Big[|1\rangle + |2\rangle\Big]$$

where $|n\rangle$ is the normalized energy eigenstate with eigenvalue $E_n = \hbar\omega(n + 1/2)$. Determine an expression for the state at any later time and use this to determine the expectation value of position measurement outcomes $\langle x \rangle$.

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Question 7

Do **either** part a) or part b) for full credit.

a) A particle in a spherically symmetric potential is in the state

$$|\psi\rangle = \frac{1}{2}\left[|1,1\rangle + \sqrt{2}\,|1,0\rangle + |1,-1\rangle\right]$$

Determine $\hat{L}_x\,|\psi\rangle$ and use this to describe as precisely as possible (this could be statistical) what a measurement of L_x will yield for a particle in this state.

Question 7 continued . . .

b) Consider the *possible* observable for a spin-1 particle

$$\hat{S} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1-i & 0 \\ 1+i & 0 & 1-i \\ 0 & 1+i & 0 \end{pmatrix}.$$

Verify that this satisfies the requirements for an observable and determine the possible outcomes of the measurements associated with this.

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