Phys 312 2021

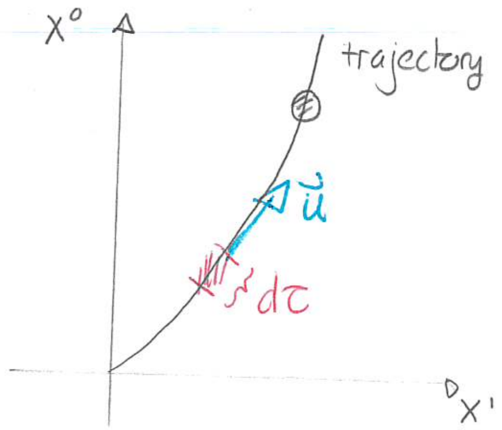
Lecture 26

Next HW Monday 10 May

Week of 10 – 14 May → classes via Zoom

Final exam – Ch 10 → 12

Velocity in relativity

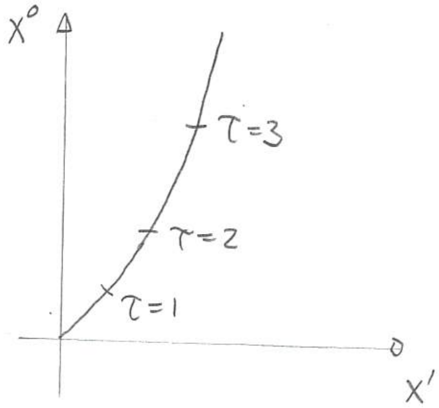


In special relativity all observers agree on the proper time for an object. The proper time is denoted τ and for infinitesimally separated events is related to the time t via

$$d\tau = dt\sqrt{1 - u^2/c^2}$$

where u is the instantaneous (ordinary) speed as observed in the frame of reference. Such proper times can be computed along the trajectory of the particle and used to parameterize the trajectory in a frame-invariant way. This gives a trajectory

$$\left(x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau)\right).$$



The four velocity is defined as:

$$\{u^\mu\} := \begin{pmatrix} \frac{dx^0}{d\tau} \\ \frac{dx^1}{d\tau} \\ \frac{dx^2}{d\tau} \\ \frac{dx^3}{d\tau} \end{pmatrix}$$

This is also called the proper velocity. Note that this gives:

$$u^0 = \frac{dx^0}{d\tau} = \frac{d\,ct}{d\tau} = c\,\frac{dt}{d\tau} = \frac{1}{\sqrt{1-u^2/c^2}}\,c$$

$$u^1 = \frac{dx^1}{d\tau} = \frac{dx^1}{dt}\frac{dt}{d\tau} = \frac{u_x}{\sqrt{1-u^2/c^2}}$$

$$\vdots$$

Thus

$$\boxed{\begin{aligned} u^0 &= \frac{1}{\sqrt{1-u^2/c^2}}\,c \\ u^1 &= \frac{1}{\sqrt{1-u^2/c^2}}\,u_x \\ u^2 &= \frac{1}{\sqrt{1-u^2/c^2}}\,u_y \\ u^3 &= \frac{1}{\sqrt{1-u^2/c^2}}\,u_z \end{aligned}}$$

ordinary velocity components

ordinary speed

These can be inverted using

$$\frac{1}{\sqrt{1-u^2/c^2}} = \frac{u^0}{c}.$$

and thus:

$$u^1 = \frac{u^0}{c} u_x \quad \Rightarrow \quad u_x = c\,\frac{u^1}{u^0}$$

$$u_y = c\,\frac{u^2}{u^0}$$

$$u_z = c\,\frac{u^3}{u^0}$$

The proper velocity transforms under the Lorentz transformations

$$x'^0 = \gamma(x^0 - \beta x^1)$$
$$x'^1 = \gamma(x^1 - \beta x^0)$$
$$x'^2 = x^2$$
$$x'^3 = x^3$$

as:

$$\boxed{\begin{aligned} u'^0 &= \gamma(u^0 - \beta u^1) \\ u'^1 &= \gamma(u^1 - \beta u^0) \\ u'^2 &= u^2 \\ u'^3 &= u^3 \end{aligned}}$$

With these rules for transformation we arrive at a situation where different observers ascribe different components to the same vector but give to co-ordinate transformations and the resulting translations they agree that the vectors are the same.

Proper momentum

The proper momentum in relativity combines the mass of a particle with the proper velocity via

$$p^\mu = m u^\mu$$

and thus can be used to construct a proper momentum four vector. This vector transforms in the same way as the proper velocity in relativity. In terms of the time and position co-ordinates used in one frame

$$p^0 = m u^0 = mc \frac{1}{\sqrt{1-u^2/c^2}}$$

$$p^1 = m u^1 = \frac{1}{\sqrt{1-u^2/c^2}} p_x$$

We note that when $u \ll c$ we get

$$p^0 c \approx mc^2 \left(1 + \frac{1}{2}\frac{u^2}{c^2}\right) = \underbrace{mc^2}_{\text{rest energy}} + \underbrace{\tfrac{1}{2}mu^2}_{\text{kinetic energy}}$$

Thus we interpret

Energy of the particle is $E = p^0 c$

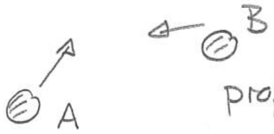
and

$$p^1 = \frac{1}{\sqrt{1-u^2/c^2}} p_x$$

$$p^2 = \frac{1}{\sqrt{1-u^2/c^2}} p_y$$

$$p^3 = \frac{1}{\sqrt{1-u^2/c^2}} p_z$$

We can consider situations where multiple particles interact



proper momentum $\{p_B{}^\mu\}$

proper momentum $\{p_A{}^\mu\}$

total proper momentum $\{P_{tot}{}^\mu\} = \{p_A{}^\mu\} + \{p_B{}^\mu\}$.

Then, if one observer finds that the total proper momentum is constant so will any other observer (although it may be a different constant)

This is the basis for <u>conservation of energy and momentum:</u>

> For any isolated system the total proper momentum $\{p_{tot}{}^\mu\}$ stays constant. All observers agree on the fact that it stays constant.

This assumption must be tested against experimental evidence which usually emanates from high energy particle physics.

Forces

Given a four-momentum we can relate this to a presumable 4-force $\{F^\mu\}$ via

$$\boxed{F^\mu = \frac{dp^\mu}{d\tau}}$$

We would then need rules specifying the components of the four-force in various situations. In electromagnetic theory the Lorentz force law will do so.

Again this 4-force transforms as $\{x^\mu\}$ does under the Lorentz transformations

We can use the 4-force to define a <u>frame-dependent relativistic force</u>

Using a frame in which t is the time co-ordinate, the relativistic force has components

$$F_x = \frac{dp^1}{dt}$$

components of proper momentum in S frame.

$$F_y = \frac{dp^2}{dt}$$

time in S frame

$$F_z = \frac{dp^3}{dt}$$

1 Ordinary force and modified Newton's Second Law

The ~~ordinary~~ force components are

relativistic

$$F_x = \frac{dp^1}{dt} \cdots$$

Show that

$$F_x = \frac{m}{\sqrt{1-u^2/c^2}} \left[a_x + \frac{1}{1-u^2/c^2} \frac{(\mathbf{u}\cdot\mathbf{a})}{c^2} u_x \right].$$

Answer:

$$p^1 = mu^1 = \frac{m}{\sqrt{1-u^2/c^2}} u_x$$

So $$\frac{dp^1}{dt} = m \left\{ \frac{du_x}{dt} \frac{1}{\sqrt{1-u^2/c^2}} + u_x \frac{d}{dt} \left(1-u^2/c^2\right)^{-1/2} \right\}.$$

$$= m \left\{ a_x \frac{1}{\sqrt{1-u^2/c^2}} + \left(-\frac{1}{2}\right)\left(-\frac{1}{c^2}\right)\left(1-u^2/c^2\right)^{-3/2} \frac{d}{dt}u^2 \, u_x \right\}.$$

Now $$u^2 = \vec{u}\cdot\vec{u}$$

$$\Rightarrow \frac{du^2}{dt} = 2\frac{d\vec{u}}{dt}\cdot\vec{u} = 2\vec{u}\cdot\vec{a}.$$ Thus.

$$\frac{dp^1}{dt} = \frac{m}{\sqrt{1-u^2/c^2}} \left[a_x - \frac{(\vec{u}\cdot\vec{a})}{c^2} \frac{1}{1-u^2/c^2} u_x \right]$$

Collecting components gives:

For a particle with ordinary velocity $\vec{u}$ and acceleration $\vec{a}$ as measured in the S frame the relativistic force as defined before gives:

$$\vec{F} = \frac{m}{\sqrt{1-u^2/c^2}} \left\{ \vec{a} + \frac{1}{1-u^2/c^2} \frac{(\vec{u}\cdot\vec{a})}{c^2} \vec{u} \right\}$$

where $u^2 = \vec{u}\cdot\vec{u}$

This is a relativistic version of Newton's Second Law. Clearly when $u \ll c$ the law reduces to the classical version of Newton's Second Law.

Force transformations

Suppose that the force is defined as

$$F_x = \frac{dp^1}{dt} \quad \dots$$

Then in the primed frame

$$F_x' = \frac{dp'^1}{dt'} \quad \dots$$

The question would be how these are related. We can show:

$$\boxed{\begin{array}{l} \text{If the frame } S' \text{ moves relative to } S \text{ with velocity } \vec{v} = v\hat{x} \\ \text{then, for a particle with velocity } \vec{u} \text{ as measured in } S\text{, the} \\ \text{relativistic force components in the two frames are related by} \\ F_x' = \left[F_x - \beta/c \, \vec{u}\cdot\vec{F}\right] / (1 - \beta/c \, u_x) \\ F_y' = \dfrac{1}{\gamma(1-\beta/c \, u_x)} F_y \\ F_z' = \dfrac{1}{\gamma(1-\beta/c \, u_x)} F_z \end{array}}$$

Proof: First consider $F_x' = \dfrac{dp'^1}{dt'}$

We need dp'^1 and dt' in terms of unprimed variables

Here $p'^1 = \gamma(p^1 - \beta p^0)$

$$\Rightarrow \quad dp'^1 = \gamma \, dp^1 - \beta\gamma \, dp^0$$

$$dt' = \frac{1}{c} dx'^0 \qquad x'^0 = \gamma(x^0 - \beta x^1)$$

$$= \frac{1}{c}\left(\gamma\, dx^0 - \gamma\beta\, dx^1\right)$$

$$= \gamma \frac{1}{c} dx^0 - \frac{\gamma\beta}{c} dx^1$$

$$= \gamma\, dt - \frac{\gamma\beta}{c} dx^1$$

Thus $$F_x' = \frac{\cancel{\gamma}(dp^1 - \beta dp^0)}{\cancel{\gamma}(dt - \beta/c\, dx^1)} = \frac{\frac{dp^1}{dt} - \frac{\beta}{c}\frac{dp^0}{dt}}{1 - \frac{\beta}{c}\frac{dx^1}{dt}}$$

But $$u_x = \frac{dx^1}{dt}$$

$$F_x = \frac{dp^1}{dt}$$

This leaves $$\frac{dp^0}{dt} = \frac{d}{dt}\, mc \frac{1}{\sqrt{1-u^2/c^2}}$$

$$= mc \frac{1}{(1-u^2/c^2)^{3/2}}\left(-\frac{1}{2}\right)\left(-\frac{1}{c^2}\right) 2\vec{u}\cdot\vec{a}$$

$$= \frac{m}{c}\frac{1}{(1-u^2/c^2)^{3/2}}\vec{u}\cdot\vec{a}$$

Then $$\vec{F} = \frac{m}{\sqrt{1-u^2/c^2}}\left\{\vec{a} + \frac{1}{1-u^2/c^2}\frac{(\vec{u}\cdot\vec{a})}{c^2}\vec{u}\right\}$$

$$\Rightarrow \vec{u}\cdot\vec{F} = \frac{m}{\sqrt{1-u^2/c^2}}\left\{(\vec{u}\cdot\vec{a})\underbrace{\left[1 + \frac{u^2/c^2}{1-u^2/c^2}\right]}_{1/1-u^2/c^2}\right\}$$

$$\Rightarrow \vec{u}\cdot\vec{F} = \frac{m}{(1-u^2/c^2)^{3/2}}\vec{u}\cdot\vec{a}$$

So

$$F_x' = \frac{F_x - \frac{\beta}{c}\vec{u}\cdot\vec{F}}{1 - \beta/c\, u_x}$$

Now consider:

$$F_y' = \frac{dp'^2}{dt'} = \frac{dp^2}{dt}\frac{dt}{dt'} = F_y \frac{dt}{dt'} = F_y \Big/ \frac{dt'}{dt}$$

But $x'^0 = \gamma x^0 - \gamma\beta x^1$

$\Rightarrow$ $t' = \gamma t - \frac{\gamma\beta}{c} x^1$

$\Rightarrow$ $\frac{dt'}{dt} = \gamma - \frac{\gamma\beta}{c} u_x = \gamma(1 - \beta/c)\, u_x$

Thus: $$F_y' = F_y \frac{1}{\gamma(1 - \frac{\beta}{c} u_x)}$$

Similarly for F_z' ...

2 Transformations of Forces in Relativity

The transformations between relativistic forces are:

$$F'_x = \frac{F_x - \frac{\beta}{c}\mathbf{u}\cdot\mathbf{F}}{1 - \frac{\beta}{c}u_x}$$

$$F'_y = \frac{F_y}{\gamma\left(1 - \frac{\beta}{c}u_x\right)}$$

$$F'_z = \frac{F_z}{\gamma\left(1 - \frac{\beta}{c}u_x\right)}$$

a) Show that if $v \ll c$, these reduce to the classical force transformations.

b) Simplify these for the case where the force is perpendicular to the particle velocity.

Answer: a) If $v \ll c$ then $\beta/c \ll 1$ and we get

$$\left.\begin{aligned} F_x' &\approx \frac{F_x}{1} = F_x \\ F_y' &\approx \frac{F_y}{1} = F_y \\ F_z' &\approx \frac{F_z}{1} = F_z \end{aligned}\right\}$$

usual transformation rules under Galilean relativity.

b) $\vec{u}\cdot\vec{F} = 0$ and

$$F_x' = \frac{1}{1-\frac{\beta u_x}{c}} F_x$$

$$F_y' = \frac{1}{\gamma\left(1-\frac{\beta u_x}{c}\right)} F_y$$

$$F_z' = \frac{1}{\gamma\left(1-\frac{\beta u_x}{c}\right)} F_z$$

3 Force exerted by an infinite line of charge on a point test charge

An infinite line of charges are at rest relative to each other along the x axis. The charge density as observed in a frame in which they are at rest is λ. A test charge with charge Q is a distance s from the line and at rest relative to the charges in the line. Let S be the frame in which these are all at rest.

a) In the S frame, determine the electric and magnetic fields produced by the line of charges and the force $\mathbf{F}$ that these exert on the test charge.

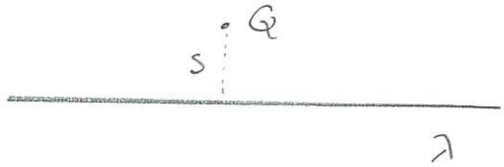
Let S' be a frame moving with velocity $\mathbf{v}$ left along the $-x$ axis.

b) In the S' frame, determine the charge density, the electric and magnetic fields produced by the line of charges and the force $\mathbf{F}'$ that these exert on the test charge.

It should be possible to calculate the force $\mathbf{F}'$ as observed in S' from $\mathbf{F}$ as observed in S using the relativistic force transformation and compare to that determined directly via the fields.

c) Use the relativistic force transformations to determine $\mathbf{F}'$ from $\mathbf{F}$. Does this agree with the result as obtained via the fields?

Answer: a) Here Gauss' Law gives



$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{y}$$

The charges are stationary so $\vec{B} = 0$

The Lorentz force law gives:

$$\vec{F} = Q(\vec{E} + \cancel{\vec{u}\times\vec{B}}^{\,0}) \Rightarrow \vec{F} = \frac{Q\lambda}{2\pi\epsilon_0 s}\hat{y}$$

b) The source charges appear to move with velocity $\vec{v} = v\hat{x}$.
The charge density is

$$\lambda' = \gamma\lambda = \frac{1}{\sqrt{1-v^2/c^2}}\lambda.$$

Then the usual rules for determining fields give:

$$\vec{E}' = \frac{\lambda'}{2\pi\epsilon_0 s}\hat{y} = \frac{\gamma\lambda}{2\pi\epsilon_0 s}\hat{y}$$

and

$$\vec{B}' = \frac{1}{c^2}\vec{v}\times\vec{E}'$$

(collection of moving point charges. $\vec{B} = \frac{1}{c^2}\vec{v}\times\vec{E}$)

$$= \frac{v}{c^2}\hat{x}\times\frac{\lambda'}{2\pi\epsilon_0 s}\hat{y}$$

$$= \frac{\gamma\lambda}{2\pi\epsilon_0 s}\frac{v}{c^2}\hat{z}$$

Then the test charge moves with velocity $\vec{u}' = \vec{v}$. So

$$\vec{F}' = Q(\vec{E}' + \vec{u}'\times\vec{B})$$

$$= Q\left\{\frac{\gamma\lambda}{2\pi\epsilon_0 s}\hat{y} + \frac{\gamma\lambda}{2\pi\epsilon_0 s}\frac{v}{c^2}v\underbrace{\hat{x}\times\hat{z}}_{-\hat{y}}\right\}$$

$$= Q\frac{\gamma\lambda}{2\pi\epsilon_0 s}\underbrace{\left(1-\frac{v^2}{c^2}\right)}_{1/\gamma^2}\hat{y}$$

$\Rightarrow \vec{F}' = \frac{1}{\gamma}Q\frac{\gamma\lambda}{2\pi\epsilon_0 s}\hat{y}$ we see that $F_y' = \frac{1}{\gamma}F_y$

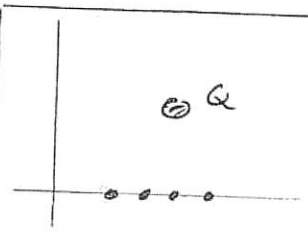
c) Here $F_y' = \dfrac{F_y}{\gamma(1-\beta/c\, u_x)}$ But $u_x = 0$. Then

$$F_y' = \frac{1}{\gamma}F_y$$

These agree. Thus the definition of the relativistic force is such that the Lorentz force law is correct in both frames.

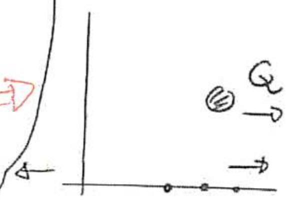
Thus we have shown that electromagnetism, for this case is invariant under Lorentz transformations:

S frame



all charges at rest. Described by $\rho, \vec{J}$

S' frame



Charge density described by $\rho', \vec{J}'$ connected via relativistic rules to $\rho, \vec{J}$

Lorentz transform

Maxwell's eqns give fields via:

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

etc,...

Maxwell's eqns give fields:

$$\vec{\nabla}' \cdot \vec{E}' = \rho'/\epsilon_0$$

etc...

Lorentz force law gives force

$$\vec{F} = Q(\vec{E} + \vec{u} \times \vec{B})$$

and

$$\vec{F} = \frac{d\vec{p}}{dt}$$ relativistic momentum

Lorentz force law gives force

$$\vec{F}' = Q(\vec{E}' + \vec{u}' \times \vec{B}')$$

and

$$\vec{F}' = \frac{d\vec{p}'}{dt'}$$

relativistic force transforms relate forces

We can either follow the black or the red arrows to relate forces