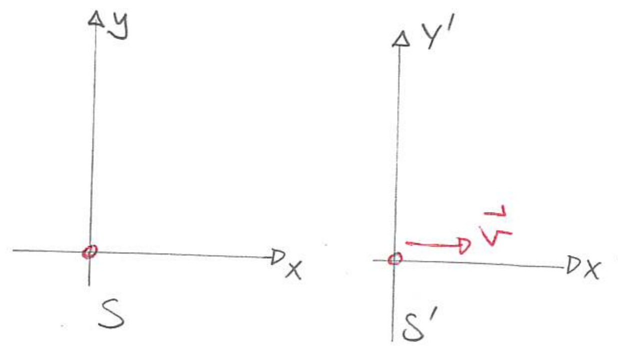
Weds: HW

Thurs: Read. 12.2.2. 12.2.4, 12.3.1

We have seen that their are issues with the laws of electromagnetism (Maxwell's equations, Lorentz Force law) and classical physics (Galilean relativity, Newton's 2nd Law).

The proposal for correcting these inconsistencies will be to assume that the co-ordinates for events as observed in two inertial frames are related by the Lorentz transformations

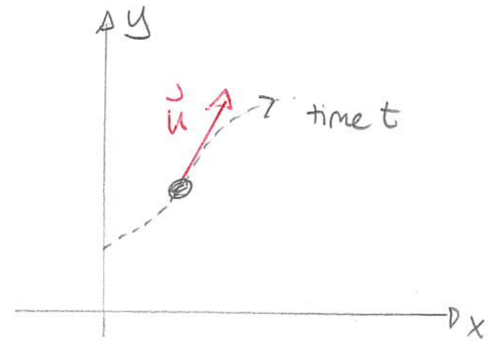
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$



where $\gamma = 1/\sqrt{1 - v^2/c^2}$. We already saw that this could affect a charge density. If charges are at rest in the primed frame with a linear charge density λ' along the x'-axis, then in the unprimed frame the charge density is $\lambda = \gamma\lambda'$.

We need to extend this to describe dynamics and forces as viewed from the different frames.

Recall that in classical physics we consider the trajectory of a particle. Then the velocity of the particle is the vector



$$\vec{u} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$$

and this has as its interpretation that it is the tangent to the trajectory. This leads to the momentum

$$\vec{p} = m\vec{u}$$

and the acceleration

$$\vec{a} = \frac{d\vec{u}}{dt}$$

Finally we reach Newton's Second Law

$$\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}.$$

The difficulty in special relativity is that time is not a parameter on which all observers agree. It will therefore yield different tangent velocity vectors and acceleration vectors when assessed in two distinct frames. We cannot expect that it will yield the same force laws.

We will have to:

1) develop a different notion for trajectories of particles

2) find a parameter on which all observers agree and use this to redefine kinematic quantities such as velocity and momentum.

3) redefine the basic force law.

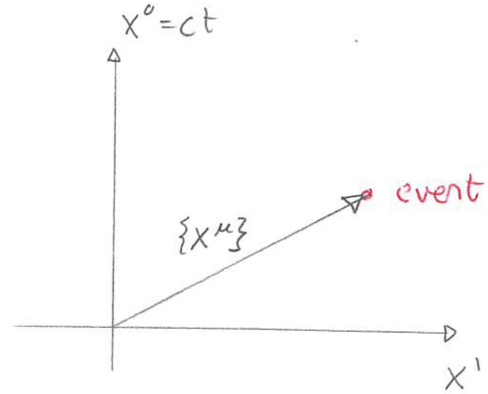
Vectors in relativity

An event in relativity consists of an occurrence associated with one time value and three spatial co-ordinates. We use these to construct four co-ordinates. In the unprimed frame:

$$x^0 := ct$$
$$x^1 := x$$
$$x^2 := y$$
$$x^3 := z$$

These give a four-vector in space time

$$\{x^\mu\} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

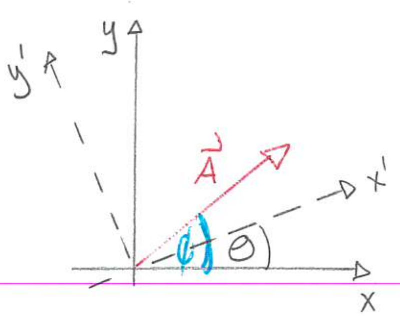


Analogy

This is analogous to a conventional two dimensional vector such as

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$

We can associate an absolute geometrical meaning with such a vector by considering how the components transform under a rotation of axes. Consider two frames that differ via a rotation. We can ask



"How can we translate between representations in the two frames so that vector operations survive the translation?"

Examples are:

1) If $\vec{A} = \begin{pmatrix} A_x \\ A_y \end{pmatrix}$ and $\vec{B} = \begin{pmatrix} B_x \\ B_y \end{pmatrix}$ in the S frame and one used these to form $\vec{C} = \vec{A} + \vec{B}$ will this translate correctly in the S′ frame as follows

S frame

$\vec{A}, \vec{B}$

form $\vec{C} = \vec{A} + \vec{B}$

rule for ~~RULE~~ translating

does this rule translate from $\vec{C}$ to $\vec{C}'$

S′ frame

$\vec{A}', B'$

form $\vec{C}' = \vec{A}' + \vec{B}'$

RULE

2) If $\vec{A} = \begin{pmatrix} A_x \\ A_y \end{pmatrix}$ and $\vec{B} = \begin{pmatrix} B_x \\ B_y \end{pmatrix}$ in the S frame we can form $\vec{A} \cdot \vec{B}$.

S frame

$\vec{A}$ $\xrightarrow{\text{Translation}}$ $\vec{A}'$

$\vec{B}$ $\xrightarrow{\text{rule}}$ $\vec{B}'$

dot product rule

$\vec{A} \cdot \vec{B}$

S′ frame

dot product rule

$\vec{A}' \cdot \vec{B}'$

do these translate

In the illustrated two dimensional situation the vectors can be written in the two frames via co-ordinates

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_x' \\ A_y' \end{pmatrix}$$

Then

$$A_x = A\cos\phi \qquad A_x' = A\cos(\phi-\theta) = \overbrace{A\cos\phi}^{A_x}\cos\theta + \overbrace{A\sin\phi}^{A_y}\sin\theta$$

$$A_y = A\sin\phi \qquad A_y' = A\sin(\phi-\theta) = \underbrace{A\sin\phi}_{A_y}\cos\theta - \underbrace{A\cos\phi}_{A_x}\sin\theta$$

Thus the translation step is:

$$\begin{pmatrix} A_x' \\ A_y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

We can easily check that if we use this to translate then the translation for addition is correct.

Addition: Given addition in one frame

$$\begin{pmatrix} C_x \\ C_y \end{pmatrix} = \begin{pmatrix} A_x \\ A_y \end{pmatrix} + \begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} A_x + B_x \\ A_y + B_y \end{pmatrix}$$

It procedes as follows in the other frame:

$$\begin{pmatrix} A_x' \\ A_y' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{\text{translation}} \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{matrix} A_x\cos\theta + A_y\sin\theta \\ -A_x\sin\theta + A_y\sin\theta \end{matrix}$$

$$B_x' = B_x\cos\theta + B_y\sin\theta$$

$$B_y' = -B_x\sin\theta + B_y\cos\theta$$

$$\Rightarrow \begin{pmatrix} A_x' \\ A_y' \end{pmatrix} + \begin{pmatrix} B_x' \\ B_y' \end{pmatrix} = \begin{pmatrix} (A_x+B_x)\cos\theta & (A_y+B_y)\sin\theta \\ -(A_y+B_y)\sin\theta & (A_y+B_y)\cos\theta \end{pmatrix}$$

add in primed frame

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_x+B_x \\ A_y+B_y \end{pmatrix}$$

$$\begin{pmatrix} C_x' \\ C_y' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{\text{same translation rule!}} \begin{pmatrix} C_x \\ C_y \end{pmatrix}$$

Dot product: Here consider

$$\vec{A} = \begin{pmatrix} A_x \\ A_y \end{pmatrix} \qquad \vec{B} = \begin{pmatrix} B_x \\ B_y \end{pmatrix}$$

Then in the unprimed frame $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y$

In the primed frame

$$\begin{pmatrix} A_x' \\ A_y' \end{pmatrix} \cdot \begin{pmatrix} B_x' \\ B_y' \end{pmatrix} = A_x'B_x' + A_y'B_y'$$

$$= (A_x\cos\theta + A_y\sin\theta)(B_x\cos\theta + B_y\sin\theta)$$

$$+ (-A_x\sin\theta + A_y\cos\theta)(-B_x\cos\theta + B_y\sin\theta)$$

$$\vdots$$

$$= A_xB_x + A_yB_y$$

$$= \vec{A}\cdot\vec{B}$$

Thus the dot product is the same in the two frames

We want the same type of results under the transformations between the co-ordinates in special relativity. We can see that

$$x'^0 = ct' = \gamma(ct - v/c\, x) = \gamma(x^0 - \beta x^1)$$

where $\beta = v/c$.

$$x'^1 = \gamma\left(x - \frac{v}{c} ct\right) = \gamma(x^1 - \beta x^0)$$

Thus the Lorentz transformations give:

Let $\{x^\mu\}$ be a 4-vector in an unprimed frame. Then in the primed frame this vector is $\{x'^\mu\}$ with
$x'^0 = \gamma(x^0 - \beta x^1)$ $\beta = v/c$
$x'^1 = \gamma(x^1 - \beta x^0)$
$x'^2 = x^2$ $\gamma = 1/\sqrt{1 - v^2/c^2}$
$x'^3 = x^3$

The translation operation can be written as

$$\underbrace{\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}}_{S' \text{ frame}} = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{translation}} \underbrace{\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}}_{S \text{ frame}}$$

We can show that just as for vectors in two-dimensional space, this translation respects addition and also scalar multiplication. We then

1) regard these vectors as geometrically identical

2) the co-ordinates in the two frames are just different representations of the same 4-vector.

We will see invariant quantities associated with the vectors. These are quantities such that

Calculate quantity in S frame

same operations $\updownarrow$

Calculate quantity in the S' frame

$\longrightarrow$ give exactly the same result.

For example in three dimensional space the length of a vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \leadsto \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

satisfies

$$L^2 = x^2+y^2+z^2 \underset{\text{same}}{\longleftrightarrow} L'^2 = (x')^2+(y')^2+(z')^2 \iff L = L'$$

This is not true in special relativity (Lorentz contraction). What can replace this as an invariant?

1 Invariant in relativity

a) Consider

$$(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

and show that this is not generally invariant.

b) Consider

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

and show that this is generally invariant.

Answer:

$$(x'^0)^2 = \gamma^2(x^0 - \beta x^1)^2 = \gamma^2\left((x^0)^2 - 2\beta x^0 x^1 + (x^1)^2\beta^2\right)$$

$$(x'^1)^2 = \gamma^2(x^1 - \beta x^0)^2 = \gamma^2\left((x^1)^2 - 2\beta x^0 x^1 + (x^0)^2\beta^2\right)$$

$$(x'^2)^2 = (x^2)^2$$

$$(x'^3)^2 = (x^3)^2$$

a) Adding gives: $\left[(x^0)^2 + (x^1)^2\right]\gamma^2(1+\beta^2) + (x^2)^2 + (x^3)^2$

$$- 4\beta\gamma^2 x^0 x^1$$

This can only equal $(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$ if $\beta\gamma^2 = 0 \Rightarrow \beta = 0$

$\Rightarrow v = 0$

b) forming $-(x'^0)^2 + (x'^1)^2 + (x'^2)^2 + (x'^3)^2$

$$= (x^0)^2\left[\gamma^2\beta^2 - \gamma^2\right] + (x^1)^2\left[\gamma^2 - \gamma^2\beta^2\right] + (x^2)^2 + (x^3)^2$$

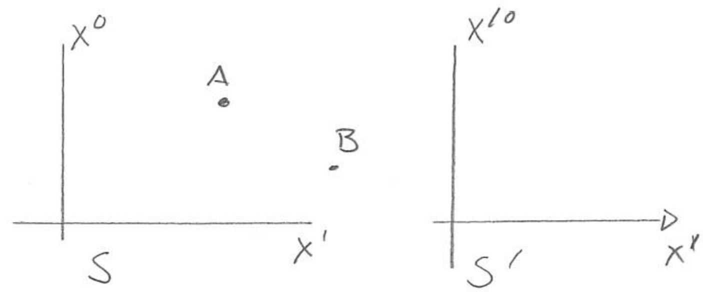
But $\gamma^2 = \frac{1}{1-\beta^2}$ and substitution gives that the expression is

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

and this is invariant.

Now consider two events A, B with co-ordinate 4-vectors:

$$\begin{pmatrix} X^0_A \\ X^1_A \\ X^2_A \\ X^3_A \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X^0_B \\ X^1_B \\ X^2_B \\ X^3_B \end{pmatrix}$$



Let $\Delta x^0 = X_B^0 - X_A^0$

$\Delta x^1 = X_B^1 - X_A^1$ etc...

Then the invariant interval is

$$\boxed{(\Delta s)^2 = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2}$$

The previous example can be use to show that:

All inertial observers obtain the same invariant interval between the same pair of events.

An infinitesimal version of this is:

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

Since all observers agree on this we will be able to use it as a parameter (rather than time) to describe trajectories and to eventually generate suitable velocity, momentum and force vectors.

Velocity in relativity

In classical physics velocity of a particle is defined via

$$\vec{u} = \frac{d\vec{r}}{dt}$$

$$\vec{u} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix}$$

We know that this will not transform correctly in special relativity since time is not invariant. How does it transform?

2 Transformation of classical velocity

Suppose that a particle moves along the x direction with constant velocity u as observed from the unprimed frame. Determine and expression for the velocity as observed from the standard primed frame.

Answer:

$$u = \frac{\Delta x}{\Delta t}$$

$$u' = \frac{\Delta x'}{\Delta t'} = \frac{\Delta x'^1}{\Delta x'^0/c} = \frac{\cancel{\gamma}(\Delta x^1 - \beta \Delta x^0)}{\cancel{\gamma}(\Delta x^0 - \beta \Delta x^1)/c}$$

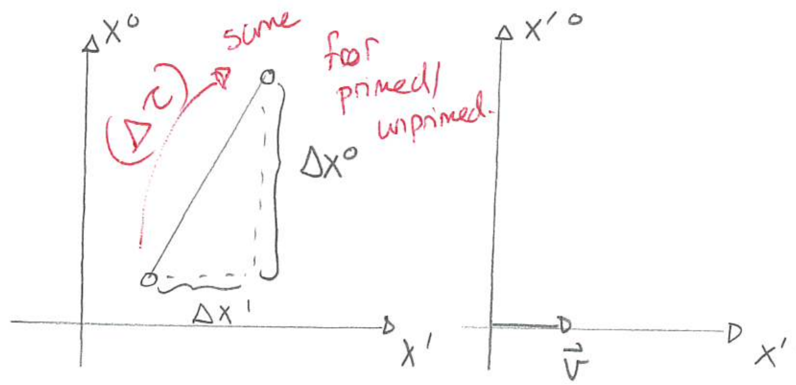
$$\Rightarrow u' = \frac{\Delta x - \frac{v}{c} c \Delta t}{\Delta t - \frac{v}{c^2} \Delta x} = \frac{\Delta x/\Delta t - v}{1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}}$$

$$\Rightarrow u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

We need to redefine velocity using rate of change of co-ordinates with respect to an invariant quantity on which all observers agree.

Consider an object whose trajectory in an x^0, x^1 graph is a straight line. Then all observers agree on

$$\Delta s^2 = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2$$



Suppose that the particle is at rest in the primed frame. Then

$$(\Delta s)^2 = -(\Delta x'^0)^2 + \underbrace{(\Delta x'^1)^2 + (\Delta x'^2)^2 + (\Delta x'^3)^2}_{=0}$$

$$= -c(\Delta t')^2$$

Clearly $(\Delta s)^2 < 0$. Then

$$(\Delta t')^2 = -\frac{(\Delta s)^2}{c^2}$$

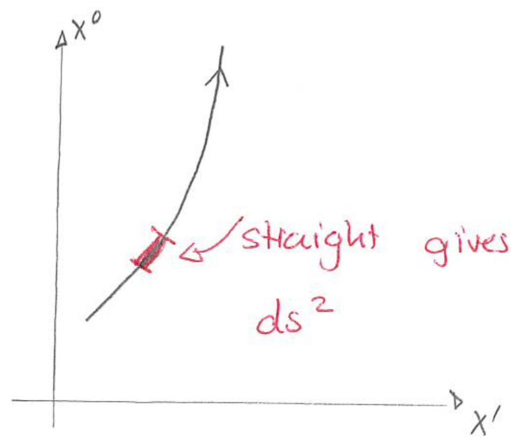
All observers agree on $-(\Delta s)^2/c^2 > 0$. Thus we define

> For a particle whose trajectory is a straight line the proper time τ is defined via
>
> $$(\Delta\tau)^2 = -\frac{(\Delta s)^2}{c^2}$$
>
> and all observers agree on this

The proper time has the interpretation that

> For a particle moving with constant velocity the proper time between events on the trajectory of the particle is the time as observed from the rest frame of the particle.

Now consider a particle which does not move with constant velocity. We can define the proper time by breaking the trajectory into infinitesimally small pieces, each approximately straight. Then along the straight segment:



$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

and

$$d\tau^2 = -ds^2/c^2$$

We can relate the proper time to co-ordinate time via:

$$d\tau^2 = -\frac{1}{c^2}\left[-c^2(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2\right]$$

$$= dt^2\left[1 - \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]\frac{1}{c^2}\right]$$

$$\Rightarrow \boxed{d\tau = dt\sqrt{1 - u^2/c^2}}$$

Here u is the instantaneous velocity of the particle as measured in the frame.

We then define the velocity 4-vector via:

$$\{u^\mu\} = \begin{pmatrix} \frac{dx^0}{d\tau} \\ \frac{dx^1}{d\tau} \\ \frac{dx^2}{d\tau} \\ \frac{dx^3}{d\tau} \end{pmatrix}$$

proper velocity

All observers will agree on this as the same vector after transformation by the Lorentz contraction.

<u>Proper velocity and conventional velocity</u>

In any given frame the conventional velocity is

$$\vec{u} = u_x \hat{x} + u_y \hat{y} + u_z \hat{z}$$

where $u_x = \frac{dx}{dt}$, $u_y = \frac{dy}{dt}$, ... How are the four-velocity components related to this?

$$u^1 = \frac{dx^1}{d\tau} = \frac{dx^1}{dt}\frac{dt}{d\tau}$$

$$= u_x \frac{dt}{d\tau} \qquad\qquad d\tau = \sqrt{1-u^2/c^2}\, dt$$

$$= u_x \frac{1}{\sqrt{1-u^2/c^2}}$$

So we get

$$
\boxed{
\begin{aligned}
u^1 &= \frac{1}{\sqrt{1-u^2/c^2}}\, u_x \\
u^2 &= \frac{1}{\sqrt{1-u^2/c^2}}\, u_y \\
u^3 &= \frac{1}{\sqrt{1-u^2/c^2}}\, u_z
\end{aligned}
}
$$

spatial parts
of proper velocity

conventional velocity