Phys 312 2021

Lecture 22

Fri: HW

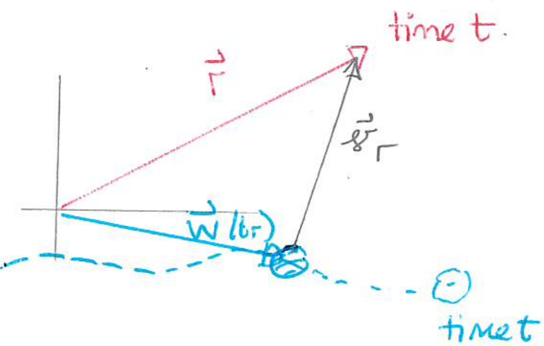
Tues: Read. 11.1.2 , 11.2.1

Fields produced by a moving point charge

We consider a point particle with charge q and trajectory $\vec{w}(t)$. The fields produced are:

1) Electric field

$$\boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}_r}{(\vec{\mathscr{r}}_r\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + \vec{\mathscr{r}}_r \times (\vec{u}\times\vec{a})\right]}$$



and the ingredients are:

a) retarded time t_r that satisfies

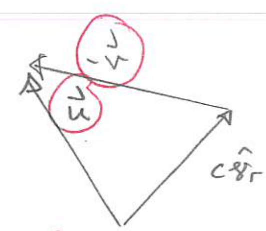
$$c(t-t_r) = |\vec{r} - \vec{w}(t_r)|$$

b) retarded separation vector

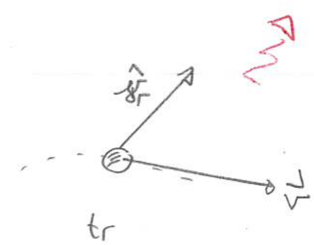
$$\vec{\mathscr{r}}_r = \vec{r} - \vec{w}(t_r) \qquad \text{magnitude} \quad \mathscr{r}_r = c(t-t_r)$$

c) velocity and acceleration

$$\vec{v} = \frac{d\vec{w}}{dt}\Big|_{t_r} \qquad \vec{a} = \frac{d^2\vec{w}}{dt^2}\Big|_{t_r}$$



d) vector $\quad \vec{u} = c\hat{\mathscr{r}}_r - \vec{v}$



2) Magnetic field

$$\boxed{\vec{B} = \frac{1}{c}\,\hat{\mathscr{r}}_r \times \vec{E}(\vec{r},t)}$$

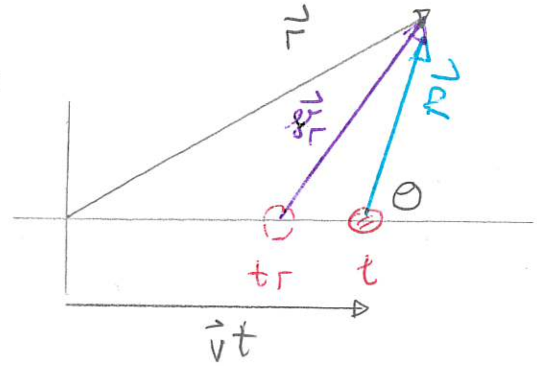
Constant velocity point source

We showed that, if the particle velocity is constant

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{1}{\mathscr{r}_r^{\,2}} \frac{1}{(1-\hat{\mathscr{r}}_r\cdot\vec{v}/c)^3} \left(1-\frac{v^2}{c^2}\right) \frac{(\vec{r}-\vec{v}t)}{\mathscr{r}_r}$$

This refers to the retarded separation vector $\vec{\mathscr{r}}$. It will be more convenient to refer to the current position of the particle. Then the current separation vector is

$$\vec{R} = \vec{r} - \vec{v}t.$$



In an exercise we will see that

$$1-\hat{\mathscr{r}}\cdot\vec{v}/c = \frac{R}{\mathscr{r}_r}\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}$$

where θ is the illustrated angle, between $\vec{v}$ and $\vec{R}$. Thus:

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{1}{R^2} \frac{1-v^2/c^2}{(1-v^2/c^2\sin^2\theta)^{3/2}} \frac{\vec{R}}{R} \Rightarrow \boxed{\vec{E} = \frac{q}{4\pi\epsilon_0 R^2} \frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}} \hat{R}}$$

Separately using

$$\vec{B} = \frac{1}{c}\hat{\mathscr{r}}_r \times \vec{E}$$

we get

$$\boxed{\vec{B} = \frac{1}{c^2}\vec{v}\times\vec{E}}$$

Derivation:

$$\vec{B} = \frac{1}{c}\frac{1}{\mathscr{r}_r}\left(\vec{\mathscr{r}}_r \times \vec{E}\right)$$

$$= \frac{1}{c}\frac{1}{\mathscr{r}_r}\left(\vec{r} - \vec{v}t_r\right)\times\vec{E}$$

$$= \frac{1}{c}\frac{1}{\mathscr{r}_r}\left(\vec{R} + \vec{v}(t-t_r)\right)\times\vec{E}$$

$$= \frac{1}{c}\frac{1}{\mathscr{r}_r}(t-t_r)\ \vec{v}\times\vec{E} \qquad \text{since} \qquad \vec{R}\times\vec{E} = 0$$

$$= \frac{1}{c^2}\frac{\cancel{(t-t_r)}}{\cancel{(t-t_r)}}\ \vec{v}\times\vec{E}$$

$\boxtimes$

1 Energy flow for a charge moving with constant velocity

A charged particle moves with constant velocity **v** to the right along the x axis.

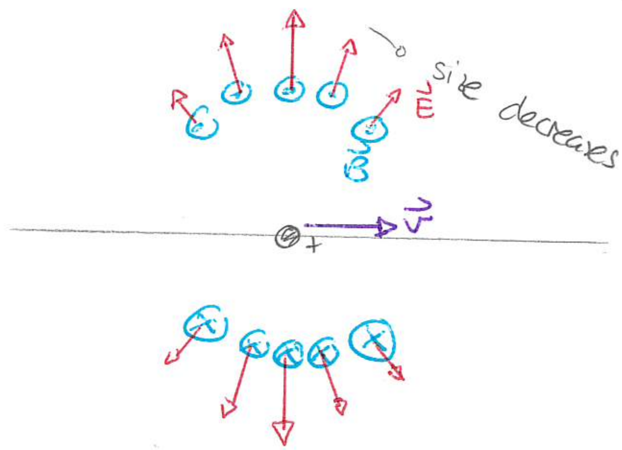
a) Sketch the fields, relative to the particle's location, at any instant. Indicate the dependence of the magnitude of the field on the angle from the x axis.

b) Determine an expression for the Poynting vector at any instant. Express this in terms of the component of **v** perpendicular to **R**.

c) Sketch the direction of energy flow at any instant.

Answer: a) The crucial term is

$$\frac{1-v^2/c^2}{(1-\frac{v^2}{c^2}\sin^2\theta)^{3/2}} \quad \begin{array}{l} \rightarrow \theta = 0 \rightarrow 1-v^2/c^2 \quad \text{smaller.} \\ \rightarrow \theta = \pi/2 \rightarrow \dfrac{1}{(1-v^2/c^2)^{1/2}} \quad \text{larger} \end{array}$$



b) $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

$$= \frac{1}{\mu_0}\frac{1}{c^2}\vec{E}\times(\vec{v}\times\vec{E}) = \frac{1}{\mu_0}\frac{1}{c^2}\left[\vec{v}(\vec{E}\cdot\vec{E}) - \vec{E}(\vec{v}\cdot\vec{E})\right]$$

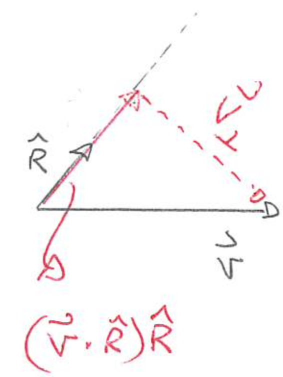
$$= \frac{1}{\mu_0}\frac{1}{c^2}\left(\frac{q}{4\pi\epsilon_0}\right)^2\frac{1}{R^4}\frac{(1-v^2/c^2)^2}{(1-v^2/c^2\sin^2\theta)^3}\left[\vec{v}(\hat{R}\cdot\hat{R}) - \hat{R}(\vec{v}\cdot\hat{R})\right]$$

$$= \frac{1}{\mu_0}\mu_0\epsilon_0\frac{q^2}{16\pi^2\epsilon_0^2}\frac{(1-v^2/c^2)^2}{(1-v^2/c^2\sin^2\theta)^3}\frac{1}{R^4}\left[\vec{v} - (\vec{v}\cdot\hat{R})\hat{R}\right]$$

Note that

$$(\vec{v}\cdot\hat{R})\,\hat{R}$$

is the component of $\vec{v}$ parallel to $\vec{R}$

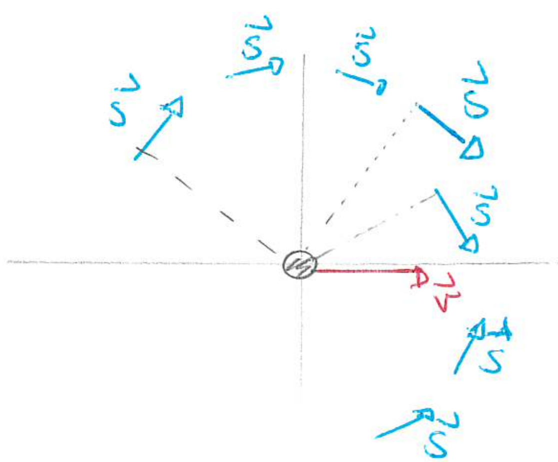


Thus

$$\vec{v} - (\vec{v}\cdot\hat{R})\,\hat{R} = \vec{v}_\perp$$

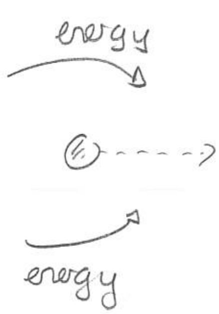
$$\Rightarrow \quad \vec{S} = \frac{q^2}{16\pi^2\epsilon_0}\,\frac{1-v^2/c^2}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^3}\,\frac{1}{R^4}\,\vec{v}_\perp$$

c)

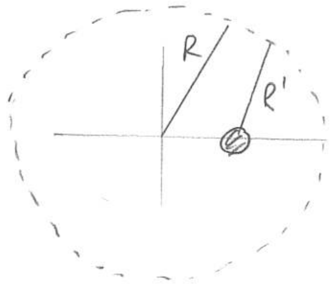


We see that the energy is flowing in the particles forwards direction

It does not appear that the particle radiates energy outwards



We can investigate radiation in the following way. Consider a sphere of radius R centered at the origin. Suppose that the particle is within the sphere. Then the total energy radiated out of the sphere is:



$$\oint \vec{S}\cdot d\vec{a}$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \vec{S}\cdot\hat{r}\; R^2 \sin\theta'\, d\theta' d\phi'$$

Then $\vec{S}\cdot\hat{r} = \frac{q^2}{16\pi^2\epsilon_0}\frac{1-v^2/c^2}{(1-v^2/c^2\sin^2\theta')^3}\frac{1}{R'^4}\vec{V}_\perp\cdot\hat{r}$. Thus the rate at which energy is radiated is:

$$\oint \vec{S}\cdot d\vec{a} = \frac{q^2}{16\pi^2\epsilon_0}\left(1-v^2/c^2\right)\int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \frac{\sin\theta'}{\left(1-\frac{v^2}{c^2}\sin^2\theta'\right)^3}\frac{R^2}{R'^4}\vec{V}_\perp\cdot\hat{r}$$

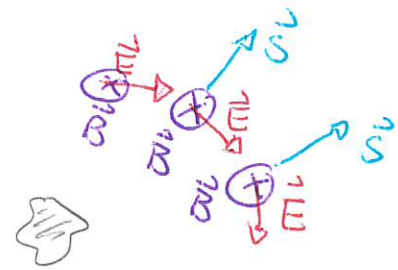
depend on θ,ϕ

Now as $R \to \infty$ $\frac{R^2}{R'^4} \to 0$ and the flux is zero. So

A particle moving with constant velocity does not radiate electromagnetic energy.

Radiation

We want to consider situations where charge distributions produce electromagnetic waves. Examples that eventually yield an outward radiating Poynting vector are:



distrib depends on t

1) oscillating electric dipoles

2) oscillating magnetic dipoles

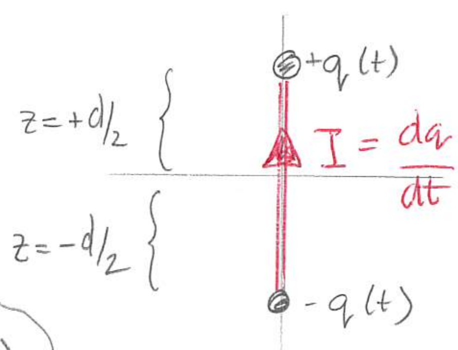
3) certain moving point charges. (accelerating).

Oscillating electric dipole

Consider a point dipole with two equal oppositely charged particles separated by distance d

There must, for charge conservation purposes, be a current connecting these.

Then the charge distribution is:



$$\rho(\vec{r}';t) = q(t)\,\delta\left(\vec{r}' - \tfrac{d}{2}\hat{z}\right) - q(t)\,\delta\left(\vec{r}' + \tfrac{d}{2}\hat{z}\right)$$

and the current density satisfies

$$\vec{J}(\vec{r}';t)\,d\tau' \begin{cases} \frac{dq}{dt}\,dz\;\hat{z} & -d/2 \le z \le d/2 \\ 0 & \text{otherwise} \end{cases}$$

We will need to compute a scalar potential

$$\boxed{V(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,', t_r)}{\mathscr{r}} \, d\tau'}$$

where $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

$t_r = t - \mathscr{r}/c$

There will also be a vector potential:

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,', t_r)}{\mathscr{r}} \, d\tau'}$$

We consider an approximate solution that uses:

1) the observation point is distant from the charges (dipole approx)

2) the oscillation rate is such that the inversion of the charge distribution would be non-relativistic

$$\boxed{\frac{d}{T/2} \ll c}$$

3) the observation point is very distant compared to the wavelength

We then reach a first approximation:

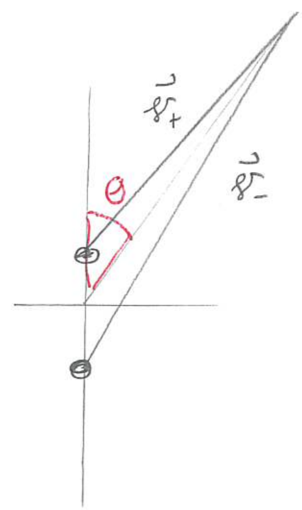
If $r \gg d$ then

$$V(\vec{r},t) \approx \frac{1}{4\pi\epsilon_0}\frac{1}{r}\left\{ q\left(t-\frac{r}{c}\left(1-\frac{d}{2r}\cos\theta\right)\right) - q\left(t-\frac{r}{c}\left(1+\frac{d}{2r}\cos\theta\right)\right)\right\}$$

$$+\frac{1}{4\pi\epsilon_0}\frac{d}{2r^2}\cos\theta\left\{ q\left(t-\frac{r}{c}\left(1-\frac{d}{2r}\cos\theta\right)\right) + q\left(t-\frac{r}{c}\left(1+\frac{d}{2r}\cos\theta\right)\right)\right\}$$

Proof: Use the illustrated geometrical scheme.

Then



$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)}{\mathscr{r}}\,d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q(t-\frac{\mathscr{r}_+}{c})}{\mathscr{r}_+} - \frac{q(t-\frac{\mathscr{r}_-}{c})}{\mathscr{r}_-}\right]$$

Then $\vec{\mathscr{r}}_\pm = \vec{r} \mp \frac{d}{2}\hat{z}$. So

$$\mathscr{r}_\pm = \sqrt{\vec{\mathscr{r}}_\pm \cdot \vec{\mathscr{r}}_\pm} = \left[r^2 + \frac{d^2}{4} \mp d\,\underbrace{\vec{r}\cdot\hat{z}}_{r\cos\theta}\right]^{1/2}$$

$$= r\left[1+\frac{d^2}{4r^2} \mp \frac{d}{r}\cos\theta\right]^{1/2}$$

$$\approx r\left[1 \mp \frac{d}{2r}\cos\theta\right]$$

So

$$V = \frac{1}{4\pi\epsilon_0}\left\{ \frac{q\left(t - \frac{r}{c}\left(1 - \frac{d}{2r}\cos\theta\right)\right)}{r\left(1 - \frac{d}{2r}\cos\theta\right)} - \frac{q\left(t - \frac{r}{c}\left(1 + \frac{d}{2r}\cos\theta\right)\right)}{r\left(1 + \frac{d}{2r}\cos\theta\right)} \right\}.$$

Now $\frac{1}{1 \pm x} \approx 1 \mp x$ for x small. So

$$V \approx \frac{1}{4\pi\epsilon_0}\frac{1}{r}\left\{ q\left(t - \frac{r}{c}\left(1 - \frac{d}{2r}\cos\theta\right)\right)\left[1 + \frac{d}{2r}\cos\theta\right] \right.$$

$$\left. - q\left(t - \frac{r}{c}\left(1 + \frac{d}{2r}\cos\theta\right)\right)\left[1 - \frac{d}{2r}\cos\theta\right] \right\}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\left\{ q\left(t - \frac{r}{c}\left(1 - \frac{d}{2r}\cos 2\theta\right)\right) - q\left(t - \frac{r}{c}\left(1 + \frac{d}{2r}\cos\theta\right)\right) \right\}$$

$$+ \frac{1}{4\pi\epsilon_0}\frac{d}{2r^2}\cos\theta\left\{ q\left(t - \frac{r}{c}\left(1 - \frac{d}{2r}\cos\theta\right)\right) + q\left(t - \frac{r}{c}\left(1 + \frac{d}{2r}\cos\theta\right)\right) \right\} \qquad \square$$

Suppose that the dipole oscillates via

$$q(t) = q_0 \cos(\omega t)$$

We now invoke another aspect of the approximation - the inversion speed must be non-relativistic. Then

time 0 — time $T/2$ (half period

$+q$

$-q$

$-q$

inverts $+q$

$\Rightarrow$ speed $= \frac{d}{(T/2)} = \frac{2d}{T} \ll c.$

Now $T = \frac{1}{f} \Rightarrow 2fd \ll c \Rightarrow 2\pi f\, d \ll c\pi$

$\Rightarrow \omega d \ll c \Rightarrow d \ll \frac{c}{\omega}$

This results in:

If $d \ll r$ and $d \ll \frac{c}{\omega}$ and $q(t) = q_0 \sin(\omega t)$

$$V(\vec{r},t) \approx \frac{1}{4\pi\epsilon_0} \frac{p_0 \cos\theta}{r} \left\{ -\frac{\omega}{c} \sin\left[\omega(t - r/c)\right] + \frac{1}{r} \cos\left[\omega(t - r/c)\right] \right\}$$

where $p_0 = q_0 d$ is the magnitude of the dipole.

We prove this in an exercise.

2 Oscillating dipole

If $r \ll d$ then the general time varying dipole yields the following approximate potential:

$$V(\mathbf{r},t) \approx \frac{1}{4\pi\epsilon_0 r}\left\{q\left[t-\frac{r}{c}\left(1-\frac{d}{2r}\cos\theta\right)\right]-q\left[t-\frac{r}{c}\left(1+\frac{d}{2r}\cos\theta\right)\right]\right\}$$

$$+\frac{d\cos\theta}{8\pi\epsilon_0 r^2}\left\{q\left[t-\frac{r}{c}\left(1-\frac{d}{2r}\cos\theta\right)\right]+q\left[t-\frac{r}{c}\left(1+\frac{d}{2r}\cos\theta\right)\right]\right\}$$

Suppose that

$$q = q_0\cos(\omega t)$$

and $d \ll c/\omega$.

a) Rewrite the terms in the potential in terms of

$$\cos[\omega(t-r/c)] \qquad \sin[\omega(t-r/c)] \qquad \cos[\omega d\cos\theta/2c] \qquad \sin[\omega d\cos\theta/2c]$$

b) Substitute, use the approximation and simplify to get

$$V(\mathbf{r},t) \approx \frac{p_0\cos\theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega}{c}\sin[\omega(t-r/c)]+\frac{1}{r}\cos[\omega(t-r/c)]\right\}.$$

Answer: a) $q\left(t-\frac{r}{c}\left(1\mp\frac{d}{2r}\cos\theta\right)\right) = q_0\cos\left[\omega t-\frac{\omega r}{c}\left(1\mp\frac{d}{2r}\cos\theta\right)\right]$

$$= q_0\cos\left[\omega t-\frac{\omega r}{c}\pm\frac{\omega d}{2c}\cos\theta\right]$$

$$= q_0\cos\left[\omega(t-r/c)\pm\omega d\cos\theta/2c\right]$$

$$= q_0\left\{\cos[\omega(t-r/c)]\cos[\omega d\cos\theta/2c] \mp \sin[\omega(t-r/c)]\sin[\omega d\cos\theta/2c]\right\}.$$

b) If $d \ll c/\omega$ then $\frac{\omega d}{2c} \ll 1$ so that argument is small

$$\cos[\omega d\cos\theta/2c] \approx 1-\frac{1}{2}\left(\frac{\omega d}{2c}\right)^2\cos^2\theta$$

$$\sin[\omega d\cos\theta/2c] \approx \frac{\omega d}{2c}\cos\theta$$

Thus

$$q\left(t-\frac{r}{c}\left(1\mp\frac{d}{2r}\cos\theta\right)\right) \approx q_0\left[\cos[\omega(t-r/c)]\left(1-\frac{1}{2}\left(\frac{\omega d}{2c}\right)^2\cos\theta\right)\right.$$

$$\left.\mp\frac{\omega d}{2c}\sin[\omega(t-r/c)]\cos\theta\right]$$

So

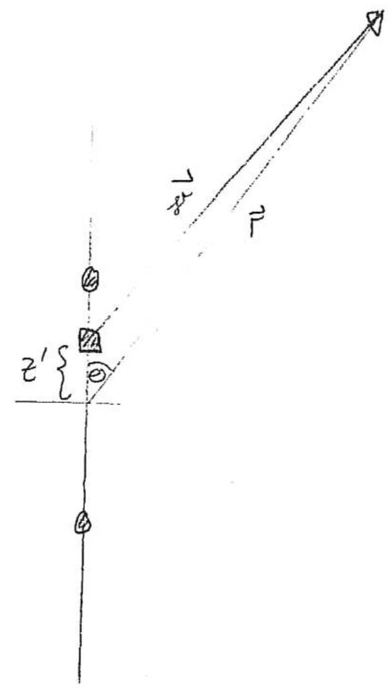
$$V \approx \frac{q_0}{4\pi\epsilon_0 r}\left\{-\cancel{2}\frac{\omega d}{\cancel{2}c}\cos\theta\ \sin[\omega(t-r/c)]\right\}$$

$$+\frac{q_0 d\cos\theta}{8\pi\epsilon_0 r^2}\left\{2\cos[\omega(t-r/c)]\right\}.$$

$$= \frac{q_0 d\cos\theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega}{c}\sin[\omega(t-r/c)]+\frac{1}{r}\cos[\omega(t-r/c)]\right\} \quad \boxtimes$$

Vector Potential

Similar calculations give the result for the vector potential.



If $r \gg d$ and $d \ll c/\omega$ then

$$\vec{A}(\vec{r},t) \approx \frac{-\mu_0 p_0 \omega}{4\pi r} \sin\left[\omega(t - r/c)\right] \hat{z}$$

Proof.

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{-d/2}^{d/2} \frac{I(\vec{r}\,',t_r)}{\mathscr{r}} \, dz' \, \hat{z}$$

and

$$\mathscr{r} = r\left[1 - 2\frac{z'}{r}\cos\theta + \left(\frac{z'}{r}\right)^2\right]^{1/2}$$

Then if $d \ll r$ and thus $z' \ll r$

$$\frac{1}{\mathscr{r}} \approx \frac{1}{r}\left[1 + \frac{z'}{r}\cos\theta\right]$$

Thus:

$$\vec{A} \approx \frac{\mu_0}{4\pi}\frac{1}{r}\int_{-d/2}^{d/2} \left(1 + \frac{z'}{r}\cos\theta\right) I(\vec{r}\,',t_r)\, dz'$$

Now $I = \dfrac{dq}{dt} = -\omega q_0 \sin(\omega t)$. Thus

$$I(\vec{r}\,',t_r) = -\omega q_0 \sin(\omega t_r)$$

$$= -\omega q_0 \sin\left[\omega(t - \mathscr{r}/c)\right]$$

$$\Rightarrow \quad I = -\omega q_0 \sin\left[\omega\left(t - \frac{r}{c}\sqrt{1 - \frac{2z'}{r}\cos\theta + \left(\frac{z'}{r}\right)^2}\right)\right]$$

Then with $\quad z' \ll r \quad$ we get.

$$I \approx -\omega q_0 \sin\left[\omega\left(t - r/c\left(1 - \frac{z'}{r}\cos\theta\right)\right)\right]$$

$$= -\omega q_0 \sin\left[\omega(t - r/c) + \frac{\omega z'}{c}\cos\theta\right]$$

$$= -q_0\omega \sin\left[\omega(t - r/c)\right]\cos\left[\frac{\omega z'}{c}\cos\theta\right]$$

$$- q_0\omega \cos\left[\omega(t - r/c)\right]\sin\left[\frac{\omega z'}{c}\cos\theta\right]$$

Then $\quad d \ll c/\omega \quad \Rightarrow \quad z' \ll c/\omega \quad$ and thus

$$I \approx -q_0\omega \sin\left[\omega(t - r/c)\right] - q_0\frac{\omega^2 z'}{c}\cos\theta\cos\left[\omega(t - r/c)\right]$$

Then $\left(1 + \frac{z'}{r}\cos\theta\right) I(\vec{r}', t_r)$ contains terms with $(z')^0, (z')^1, (z')^2, \ldots$

Then z' terms integrate to zero. Thus:

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi}\frac{1}{r}\left\{-q_0\omega \sin\left[\omega(t - r/c)\right]\right\}\int_{-d/2}^{d/2} dz' \quad \hat{z}$$

$$+ \frac{\mu_0}{4\pi}\frac{1}{r}\left\{-\frac{q_0\omega^2}{c}\cos\theta\cos\left[\omega(t - r/c)\right]\right\}\frac{1}{r}\int_{-d/2}^{d/2} z'^2 \, dz' \; \hat{z}$$

$$= -\frac{\mu_0 q_0 d\omega}{4\pi r}\sin\left[\omega(t - r/c)\right]\hat{z} - \underbrace{\frac{\mu_0 q_0}{4\pi r^2}\frac{\omega^2}{c}\cos\theta\cos\left[\omega(t - r/c)\right]\frac{2}{3}\left(\frac{d}{2}\right)^3\hat{z}}_{\text{contains } \frac{\omega d}{c} \ll 1}$$

$$\vec{A}(\vec{r}, t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin\left[\omega(t - r/c)\right]\hat{z} \qquad \blacksquare \qquad \approx 0$$