

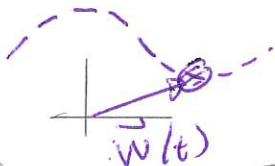
Tues: HW 5pm

Thurs: Read Ex 10.4, 11.1.1, 11.1.2.

Liénard Wiechert Potentials

The potential formalism of electromagnetism as applied to a moving point charge particle eventually gives the framework:

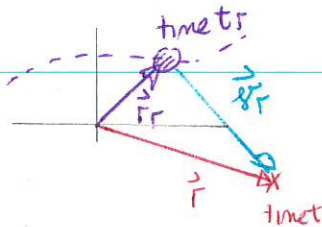
Particle with charge q and trajectory $\vec{w}(t)$



Want field at location $\vec{r}$ at time t . Determine retarded time via

$$c(t - t_r) = |\vec{r} - \vec{w}(t_r)|$$

Determine retarded position $\vec{r}_r = \vec{w}(t_r)$ and retarded separation vector $\vec{s}_r = \vec{r} - \vec{r}_r$



Scalar potential

$$V(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{c}{s_r c - \vec{s}_r \cdot \vec{v}} \quad \text{with} \quad \vec{v} = \frac{d\vec{w}}{dt} \Big|_{t_r}$$

Vector potential

$$\vec{A}(\vec{r}, t) = \frac{\vec{v}}{c^2} V(\vec{r}, t) = \frac{q}{4\pi\epsilon_0 c} \frac{\vec{v}}{s_r c - \vec{s}_r \cdot \vec{v}}$$

Liénard-Weichert
potentials.

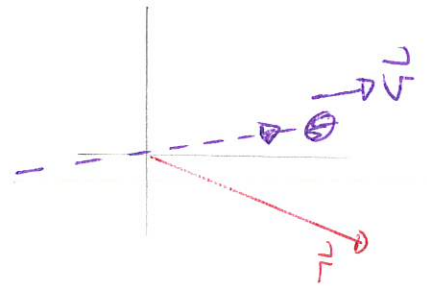
Particle moving with constant velocity

Consider the potentials produced by a particle moving with constant velocity $\vec{v}$. To simplify the discussion assume that the particle passes through the origin at $t=0$. Then

$$\vec{w} = \vec{v}t$$

We first obtain the retarded time and show

$$t_r = \frac{c^2 t - \vec{r} \cdot \vec{v} - \sqrt{(c^2 t - \vec{r} \cdot \vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}{c^2 - v^2}$$



Derivation: $c(t - t_r) = |\vec{r} - \vec{w}(t_r)|$

$$= |\vec{r} - \vec{v}t_r|$$

$$\Rightarrow c^2(t - t_r)^2 = |\vec{r} - \vec{v}t_r|^2 = (\vec{r} - \vec{v}t_r) \cdot (\vec{r} - \vec{v}t_r)$$

$$\Rightarrow c^2[t^2 - 2tt_r + t_r^2] = r^2 - v^2 t_r^2 - 2\vec{v} \cdot \vec{r} t_r$$

$$\Rightarrow t_r^2 [c^2 - v^2] + t_r [-2ct + 2\vec{v} \cdot \vec{r}] + c^2 t^2 - r^2 = 0$$

$$\Rightarrow t_r = \frac{2ct - 2\vec{v} \cdot \vec{r} \pm \sqrt{4(c^2 t - \vec{v} \cdot \vec{r})^2 - 4(c^2 t^2 - r^2)(c^2 - v^2)}}{2(c^2 - v^2)}$$

$$= \frac{(c^2 t - \vec{v} \cdot \vec{r}) \pm \sqrt{(c^2 t - \vec{v} \cdot \vec{r})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}{c^2 - v^2}$$

By the previous argument only the $-$ sign gives $t_r < t$. $\square$

We need this to calculate $\vec{s}_r$ and s_r . We get

$$\vec{s}_r = \vec{r} - \vec{w}(t_r) = \vec{r} - \vec{v} t_r$$

and

$$s_r = |\vec{r} - \vec{w}(t_r)| = c(t - t_r)$$

Then

$$s_r c - \vec{v} \cdot \vec{s}_r = \sqrt{(c^2 t - \vec{v} \cdot \vec{r})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}$$

Derivation:

$$\vec{v} \cdot \vec{s}_r = \vec{v} \cdot \vec{r} - v^2 t_r$$

$$s_r c - \vec{v} \cdot \vec{s}_r = c^2(t - t_r) - \vec{v} \cdot \vec{r} + v^2 t_r$$

$$= c^2 t - \vec{v} \cdot \vec{r} + t_r(v^2 - c^2)$$

$$= c^2 t - \vec{v} \cdot \vec{r} - (c^2 t - \vec{v} \cdot \vec{r}) + \sqrt{(c^2 t - \vec{v} \cdot \vec{r})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}$$



Thus

$$V(\vec{r}, t) = \frac{q c}{4\pi\epsilon_0} \frac{1}{\sqrt{(c^2 t - \vec{v} \cdot \vec{r})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}$$

Point source
Constant $\vec{v}$

$$\vec{A}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0 c} \frac{\vec{v}}{\sqrt{(c^2 t - \vec{v} \cdot \vec{r})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}$$

$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

$$\Rightarrow \epsilon_0 = \frac{1}{\mu_0 c^2}$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{q c \vec{v}}{\sqrt{(c^2 t - \vec{v} \cdot \vec{r})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}$$

Point source
Constant $\vec{v}$

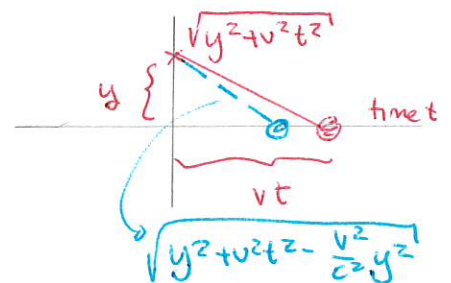
1 Potentials for a particle moving with constant velocity.

A charged particle travels with constant velocity along the $+x$ axis, passing the origin at $t = 0$.

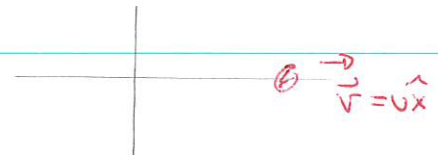
- Determine the scalar and vector potentials at any point along the y axis.
- Determine the potentials for the case where the velocity is zero. Are these consistent with those obtained by electrostatics and magnetostatics?
- Determine the potentials for the case where $v \ll c$.

Answer: a)
$$\begin{aligned} \vec{r} &= v \hat{x} \\ \vec{r} &= y \hat{y} \end{aligned} \quad \left. \vphantom{\begin{aligned} \vec{r} &= v \hat{x} \\ \vec{r} &= y \hat{y} \end{aligned}} \right\} \Rightarrow \vec{v} \cdot \vec{r} = 0$$

$$\begin{aligned} V(\vec{r}, t) &= \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{c^4 t^2 + (c^2 - v^2)(r^2 - c^2 t^2)}} \\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(c^2 - v^2)r^2 + v^2 c^2 t^2}} \\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(1 - v^2/c^2)y^2 + v^2 t^2}} \end{aligned}$$



$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{q v \hat{x}}{\sqrt{(1 - v^2/c^2)y^2 + v^2 t^2}}$$



b) $V(\vec{r}, t) = \frac{q}{4\pi\epsilon_0 |y|}$ Yes

$\vec{A}(\vec{r}, t) = 0$ Yes.

c) $V(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{y^2 + v^2 t^2}}$ } $\rightarrow$ distance to current particle location
~ electrostatic

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{q v \hat{x}}{\sqrt{y^2 + v^2 t^2}}$$

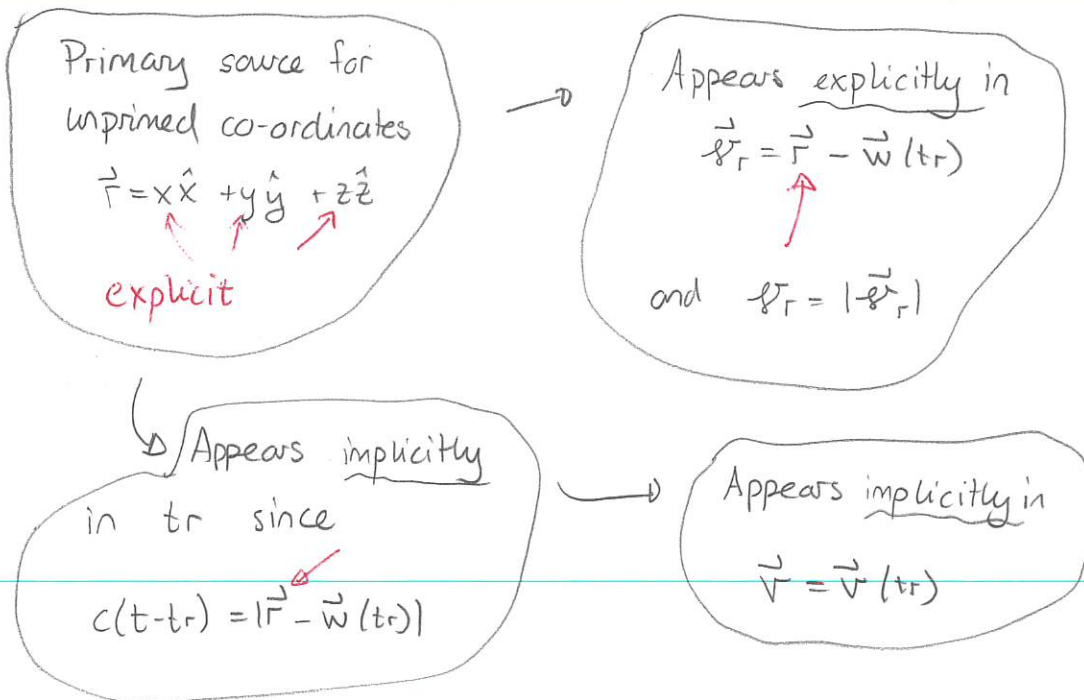
Fields produced by a moving point charge particle.

Given the potentials produced by a moving point charge we can calculate the fields by

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

where the spatial derivatives are w.r.t unprimed co-ordinates. We need to identify terms in which these appear in the potentials. These are:



Each of these: $\vec{r}$, $\vec{r}_r$, r_r , $\vec{v}$ is therefore not necessarily constant when differentiation with respect to x, y, z arises.

The relevant derivatives appear as follows (Proofs later)

Lemma 1

$$\vec{\nabla}_{tr} = \frac{-\vec{r}_r}{r_c - \vec{v} \cdot \vec{r}_r}$$

Lemma 2

$$\vec{\nabla}(\vec{r}_r \cdot \vec{v}) = \vec{v} + (\vec{r}_r \cdot \vec{a} - v^2) \vec{\nabla}_{tr}$$

where the retarded acceleration is

$$\vec{a} = \left. \frac{d^2 \vec{w}}{dt^2} \right|_{tr}$$

Lemma 3

$$\frac{\partial t_r}{\partial t} = \frac{r_c}{c r_c - \vec{v} \cdot \vec{r}_r}$$

Lemma 4

$$\frac{\partial}{\partial t}(\vec{v} \cdot \vec{r}_r) = \frac{\partial t_r}{\partial t}(\vec{r}_r \cdot \vec{a} - v^2)$$

Lemma 1: $\vec{\nabla} t_r = \frac{-\vec{r}_r}{r_r (c - \vec{r}_r \cdot \vec{v})}$

Proof: The retarded time depends implicitly on $\vec{r}$. Specifically

$$c(t - t_r) = r_r$$

$$\Rightarrow c(t - t_r) = (\vec{r}_r \cdot \vec{r}_r)^{1/2}$$

$$\Rightarrow \vec{\nabla} (c(t - t_r)) = \vec{\nabla} (\dots)^{1/2}$$

$$\Rightarrow -c \vec{\nabla} t_r = \frac{1}{2} \frac{1}{[\vec{r}_r \cdot \vec{r}_r]^{1/2}} \vec{\nabla} (\vec{r}_r \cdot \vec{r}_r)$$

$$\Rightarrow -c \vec{\nabla} t_r = \frac{1}{2 r_r} \left\{ \vec{r}_r \times (\vec{\nabla} \times \vec{r}_r) + \vec{r}_r \times (\vec{\nabla} \times \vec{r}_r) + (\vec{r}_r \cdot \vec{\nabla}) \vec{r}_r + (\vec{r}_r \cdot \vec{\nabla}) (\vec{r}_r) \right\}$$

$$\Rightarrow -c \vec{\nabla} t_r = \frac{1}{r_r} \left[\vec{r}_r \times (\vec{\nabla} \times \vec{r}_r) + (\vec{r}_r \cdot \vec{\nabla}) \vec{r}_r \right]$$

We will show:

$$\textcircled{A}: \vec{\nabla} \times \vec{r}_r = \vec{v} \times \vec{\nabla} t_r$$

$$\textcircled{B}: (\vec{r}_r \cdot \vec{\nabla}) \vec{r}_r = \vec{r}_r - \vec{v} [\vec{r}_r \cdot \vec{\nabla} t_r]$$

First to prove $\textcircled{A}$:

$$\begin{aligned} \vec{\nabla} \times \vec{r}_r &= \vec{\nabla} \times (\vec{r} - \vec{w}(t_r)) \\ &= \vec{\nabla} \times \vec{r} - \vec{\nabla} \times \vec{w}(t_r) \\ &= -\vec{\nabla} \times \vec{w}(t_r) \end{aligned}$$

But
$$\vec{\nabla} \times \vec{W} = \left(\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y} \right) \hat{z} + \left(\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z} \right) \hat{x} + \left(\frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x} \right) \hat{y}$$

and
$$\begin{aligned} \frac{\partial w_y}{\partial x} &= \frac{\partial w_y}{\partial t_r} \frac{\partial t_r}{\partial x} \\ &= \left. \frac{dw_y}{dt} \right|_{t_r} \frac{\partial t_r}{\partial x} \quad \text{etc.,...} \end{aligned}$$

gives:

$$\begin{aligned} \vec{\nabla} \times \vec{W} &= \left(\left. \frac{dw_y}{dt} \right|_{t_r} \frac{\partial t_r}{\partial x} - \left. \frac{dw_x}{dt} \right|_{t_r} \frac{\partial t_r}{\partial y} \right) \hat{z} + \dots \\ &= \left[\left. \frac{dw_y}{dt} \right|_{t_r} (\vec{\nabla} t_r)_x - \left. \frac{dw_x}{dt} \right|_{t_r} (\vec{\nabla} t_r)_y \right] \hat{z} + \dots \\ &= \vec{\nabla}(t_r) \times \left. \frac{d\vec{W}}{dt} \right|_{t_r} \end{aligned}$$

$$\Rightarrow \vec{\nabla} \times \vec{r} = -\vec{\nabla}(t_r) \times \vec{r} = \vec{r} \times \vec{\nabla}(t_r)$$

and this proves A

To prove (B): $(\vec{r} \cdot \vec{\nabla}) \vec{r} = \vec{r} \cdot \vec{\nabla} (\vec{r} - \vec{W}(t_r)).$

Then for any vector $\vec{a}$ it is easily shown that $(\vec{a} \cdot \vec{\nabla}) \vec{r} = \vec{a}.$

Thus

$$(\vec{r} \cdot \vec{\nabla}) \vec{r} = \vec{r} - \vec{r} \cdot \vec{\nabla} (\vec{W}(t_r))$$

Now

$$\begin{aligned} \vec{b} \cdot \vec{\nabla} (\vec{W}(t_r)) &= b_x \frac{\partial}{\partial x} \vec{W}(t_r) + b_y \frac{\partial}{\partial y} \vec{W}(t_r) + b_z \frac{\partial}{\partial z} \vec{W}(t_r) \\ &= b_x \left. \frac{d\vec{W}}{dt} \right|_{t_r} \frac{\partial t_r}{\partial x} + b_y \left. \frac{d\vec{W}}{dt} \right|_{t_r} \frac{\partial t_r}{\partial y} + \dots \\ &= \left. \frac{d\vec{W}}{dt} \right|_{t_r} [\vec{b} \cdot \vec{\nabla}(t_r)]. \end{aligned}$$

Thus.

$$\begin{aligned}(\vec{g}_r \cdot \vec{\nabla}) \vec{g}_r &= \vec{g}_r - \frac{d\vec{w}}{dt}\bigg|_{t_r} \vec{g}_r \cdot \vec{\nabla}(t_r) \\&= \vec{g}_r - \vec{v} (\vec{g}_r \cdot \vec{\nabla}(t_r))\end{aligned}$$

which proves B. So

$$-c \vec{\nabla}(t_r) = \frac{1}{g_r} \left\{ \vec{g}_r \times [\vec{v} \times \vec{\nabla}(t_r)] + (\vec{g}_r \cdot \vec{\nabla}) \vec{g}_r \right\}.$$

$$= \frac{1}{g_r} \left\{ \vec{v} (\vec{g}_r \cdot \vec{\nabla}(t_r)) - \vec{\nabla}(t_r) (\vec{v} \cdot \vec{g}_r) + \vec{g}_r - \vec{v} (\vec{g}_r \cdot \vec{\nabla}(t_r)) \right\}$$

$$\Rightarrow c \vec{\nabla}(t_r) = \frac{1}{g_r} \left\{ \vec{\nabla}(t_r) (\vec{v} \cdot \vec{g}_r) - \vec{g}_r \right\}$$

$$\Rightarrow \vec{\nabla}(t_r) [g_r c - \vec{v} \cdot \vec{g}_r] = -\vec{g}_r$$

$$\Rightarrow \vec{\nabla}(t_r) = \frac{-\vec{g}_r}{g_r c - \vec{v} \cdot \vec{g}_r}$$



lemma 2 $\vec{\nabla}(\vec{r}_r \cdot \vec{v}) = \vec{v} + (\vec{r}_r \cdot \vec{a} - v^2) \vec{\nabla}(t_r)$

where

$$\vec{a} = \left. \frac{d^2 \vec{w}}{dt^2} \right|_{t_r}$$

Proof. $\vec{\nabla}(\vec{r}_r \cdot \vec{v}) = \vec{r}_r \times (\vec{\nabla} \times \vec{v}) + \vec{v} \times (\vec{\nabla} \times \vec{r}_r)$
 $\uparrow$
 evaluated at t_r $+ (\vec{r}_r \cdot \vec{\nabla}) \vec{v} + (\vec{v} \cdot \vec{\nabla}) \cdot \vec{r}_r$

We saw in lemma 1 that

$$\vec{\nabla} \times \vec{r}_r = \vec{v} \times \vec{\nabla}(t_r)$$

Then $\vec{\nabla} \times \vec{v} = \left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y} \right) \hat{x} + \dots$
 $= \left(\left. \frac{dv_y}{dt} \right|_{t_r} \frac{\partial t_r}{\partial z} - \left. \frac{dv_z}{dt} \right|_{t_r} \frac{\partial t_r}{\partial y} \right) \hat{x} + \dots$
 $= [a_y (\vec{\nabla} t_r)_z - a_z (\vec{\nabla} t_r)_y] \hat{x} + \dots$
 $= \vec{\nabla}(t_r) \times \vec{a}$

So $\vec{\nabla}(\vec{r}_r \cdot \vec{v}) = \vec{r}_r \times [\vec{\nabla}(t_r) \times \vec{a}] + \vec{v} \times [\vec{v} \times \vec{\nabla}(t_r)]$
 $+ (\vec{r}_r \cdot \vec{\nabla}) \vec{v} + (\vec{v} \cdot \vec{\nabla}) \cdot \vec{r}_r$

Then $(\vec{r}_r \cdot \vec{\nabla}) \vec{v} = r_{rx} \frac{\partial \vec{v}}{\partial x} + r_{ry} \frac{\partial \vec{v}}{\partial y} + \dots$
 $= r_{rx} \left. \frac{d\vec{v}}{dt} \right|_{t_r} \frac{dt_r}{dx} + \dots$
 $= [\vec{r}_r \cdot \vec{\nabla}(t_r)] \vec{a}$

Also: $(\vec{v} \cdot \vec{\nabla}) \vec{r} = v_x \frac{\partial}{\partial x} \vec{r} + v_y \frac{\partial}{\partial y} \vec{r} + \dots$ and $\vec{r} = \vec{r} - \vec{\omega}(t) \cdot \vec{r}$

$$= v_x \hat{x} - [\vec{v} \cdot \vec{\nabla}(t)] v_x \hat{x} + \dots$$

$$= \vec{v} - [\vec{v} \cdot \vec{\nabla}(t)] \vec{v}$$

Thus: $\vec{\nabla}(\vec{r} \cdot \vec{v}) = [\vec{\nabla}(t)] \vec{r} \cdot \vec{a} - \vec{a} [\vec{r} \cdot \vec{\nabla}(t)]$

$$+ \vec{v} [\vec{r} \cdot \vec{\nabla}(t)] - \vec{\nabla}(t) v^2$$

$$+ \vec{a} [\vec{r} \cdot \vec{\nabla}(t)] + \vec{v} - \vec{v} [\vec{v} \cdot \vec{\nabla}(t)]$$

$$= \vec{v} + \vec{\nabla}(t) [\vec{r} \cdot \vec{a} - v^2]$$

This proves the result.



Lemma 3

$$c(t-t_r) = |\vec{r} - \vec{w}(t_r)|$$

$$\Rightarrow c^2(t-t_r)^2 = r^2 + w^2(t_r) - 2\vec{r} \cdot \vec{w}(t_r)$$

$$\text{Differentiate w.r.t. } t \Rightarrow c^2(t-t_r) \left[1 - \frac{\partial t_r}{\partial t} \right]$$

$$= \cancel{r} w \frac{dw}{dt} \Big|_{t_r} \frac{\partial t_r}{\partial t} - \cancel{r} \vec{r} \cdot \frac{d\vec{w}}{dt} \Big|_{t_r} \frac{\partial t_r}{\partial t}$$

$$\Rightarrow c^2(t-t_r) = \frac{\partial t_r}{\partial t} \left[c^2(t-t_r) + \vec{w} \cdot \vec{v} - \vec{r} \cdot \vec{v} \right]$$

$$\Rightarrow c \cancel{r} = \frac{\partial t_r}{\partial t} \left[c^2(t-t_r) - \vec{v} \cdot \vec{r} \right]$$

$$\Rightarrow \frac{\partial t_r}{\partial t} = \frac{c \cancel{r}}{c \cancel{r} - \vec{v} \cdot \vec{r}} \quad \square$$

Lemma 4

$$\frac{\partial}{\partial t} (\vec{v} \cdot \vec{r}) = \underbrace{\frac{\partial \vec{v}}{\partial t} \cdot \vec{r}} + \vec{v} \cdot \frac{\partial \vec{r}}{\partial t}$$

$$\underbrace{\frac{\partial \vec{v}}{\partial t_r} \frac{\partial t_r}{\partial t}}_{\vec{a}}$$

$$= \vec{a} \cdot \vec{r} \frac{\partial t}{\partial t_r} + \vec{v} \cdot \left(-\frac{\partial \vec{w}}{\partial t} \right)$$

$$= \left(\vec{a} \cdot \vec{r} - \vec{v} \cdot \vec{v} \right) \frac{\partial t}{\partial t_r}$$

$$= \left(\vec{r} \cdot \vec{a} - v^2 \right) \frac{\partial t}{\partial t_r} \quad \square$$

Electric field

Consider first

$$\begin{aligned}\vec{\nabla} V &= \frac{q_c}{4\pi\epsilon_0} \vec{\nabla} (r_c - \vec{v} \cdot \vec{r})^{-1} \\ &= -\frac{q_c}{4\pi\epsilon_0} (r_c - \vec{v} \cdot \vec{r})^{-2} [c \vec{\nabla} r_c - \vec{\nabla} (\vec{v} \cdot \vec{r})]\end{aligned}$$

$$\begin{aligned}\text{Then } \vec{\nabla} r_c &= \vec{\nabla} c(t - t_r) \\ &= -c \vec{\nabla} t_r = \frac{c \vec{r}}{r_c - \vec{v} \cdot \vec{r}}\end{aligned}$$

$$\begin{aligned}\Rightarrow \vec{\nabla} V &= -\frac{q_c}{4\pi\epsilon_0} \frac{1}{(r_c - \vec{v} \cdot \vec{r})^2} \left[\frac{c^2 \vec{r}}{r_c - \vec{v} \cdot \vec{r}} - \vec{\nabla} (\vec{v} \cdot \vec{r}) \right] \\ &= -\frac{q_c}{4\pi\epsilon_0} \frac{1}{(r_c - \vec{v} \cdot \vec{r})^2} \left[\frac{c^2 \vec{r}}{r_c - \vec{v} \cdot \vec{r}} - \vec{v} + \frac{\vec{r} (\vec{r} \cdot \vec{a} - v^2)}{(r_c - \vec{v} \cdot \vec{r})} \right] \\ &= \frac{q_c}{4\pi\epsilon_0} \frac{1}{(r_c - \vec{v} \cdot \vec{r})^3} \left[\vec{v} (r_c - \vec{v} \cdot \vec{r}) - c^2 \vec{r} + v^2 \vec{r} - \vec{r} (\vec{r} \cdot \vec{a}) \right] \\ &= \frac{q_c}{4\pi\epsilon_0} \frac{1}{(r_c - \vec{v} \cdot \vec{r})^3} \left[\vec{r} (v^2 - c^2 - \vec{r} \cdot \vec{a}) + \vec{v} (r_c - \vec{v} \cdot \vec{r}) \right]\end{aligned}$$

Second

$$\begin{aligned}\frac{\partial \vec{A}}{\partial t} &= \frac{\mu_0 q_c}{4\pi} \frac{\partial}{\partial t} \frac{\vec{v}}{r_c - \vec{r} \cdot \vec{v}} \\ &= \frac{\mu_0 q_c}{4\pi} \frac{\frac{\partial \vec{v}}{\partial t} (r_c - \vec{r} \cdot \vec{v}) - \vec{v} \frac{\partial}{\partial t} (r_c - \vec{r} \cdot \vec{v})}{(r_c - \vec{r} \cdot \vec{v})^2}\end{aligned}$$

Now
$$\frac{\partial \vec{v}}{\partial t} = \frac{\partial \vec{v}}{\partial t_r} \frac{\partial t_r}{\partial t} = \vec{a} \frac{\partial t_r}{\partial t}$$

$$\begin{aligned} \frac{\partial}{\partial t} r c &= c \frac{\partial r}{\partial t} = c \frac{\partial}{\partial t} c(t - t_r) \\ &= c^2 \left[1 - \frac{\partial t_r}{\partial t} \right] \end{aligned}$$

etc, ...

Combining these gives:

$$\frac{\partial \vec{A}}{\partial t} = \frac{q_r c}{4\pi\epsilon_0} \frac{1}{(r c - \vec{r} \cdot \vec{v})^3} \left\{ (r c - \vec{r} \cdot \vec{v}) \left(\frac{\vec{r} \cdot \vec{a}}{c} - \vec{v} \right) + \frac{\vec{r}}{c} (c^2 - v^2 + \vec{r} \cdot \vec{a}) \vec{v} \right\}$$

Now

$$\vec{\nabla} V + \frac{\partial \vec{A}}{\partial t} = \frac{q_r c}{4\pi\epsilon_0} \frac{1}{(r c - \vec{r} \cdot \vec{v})^3} \left\{ (r c - \vec{r} \cdot \vec{v}) \left(\frac{\vec{r} \cdot \vec{a}}{c} \right) + (c^2 - v^2 + \vec{r} \cdot \vec{a}) \left(\frac{\vec{r}}{c} \vec{v} - \vec{r} \right) \right\}$$

$$\Rightarrow \vec{E} = \frac{-q_r}{4\pi\epsilon_0} \frac{1}{(r c - \vec{r} \cdot \vec{v})^3} \left\{ \vec{r} \cdot \vec{a} (r c - \vec{r} \cdot \vec{v}) + (c^2 - v^2 + \vec{r} \cdot \vec{a}) (\vec{r} \vec{v} - c \vec{r}) \right\}$$

We let

$$\vec{u} = c \hat{r} - \vec{v}$$

Thus

$$\vec{E} = \frac{-q_r}{4\pi\epsilon_0} \frac{1}{(\vec{u} \cdot \vec{r})^3} \left\{ \vec{r} \cdot \vec{a} (\vec{r} \cdot \vec{u}) + (c^2 - v^2 + \vec{r} \cdot \vec{a}) \vec{r} \cdot \vec{u} \right\}$$

$$\begin{aligned} &= \frac{q_r}{4\pi\epsilon_0} \frac{1}{(\vec{u} \cdot \vec{r})^3} \left\{ (c^2 - v^2) \vec{r} \cdot \vec{u} - \underbrace{\left[\vec{r} \cdot \vec{a} (\vec{r} \cdot \vec{u}) - (\vec{r} \cdot \vec{a}) \vec{r} \cdot \vec{u} \right]}_{\vec{r} \cdot \vec{r} \times (\vec{a} \times \vec{u})} \right\} \end{aligned}$$

This gives:

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\hat{r}_r}{(\vec{u} \cdot \hat{r}_r)^3} \left\{ (c^2 - v^2) \vec{u} + \hat{r}_r \times (\vec{u} \times \vec{a}) \right\}$$

Now using similar vector calculus for $\vec{B} = \vec{\nabla} \times \vec{A}$ we get

$$\vec{B} = \frac{1}{c} (\hat{r}_r \times \vec{E})$$

Thus the process is:

Given $\vec{w}(t)$

Determine retarded time via
 $c(t - t_r) = |\vec{r} - \vec{w}(t_r)|$

Determine retarded separation vector
 $\vec{r}_r = \vec{r} - \vec{w}(t_r)$

Determine

$$\vec{v} = \left. \frac{d\vec{w}}{dt} \right|_{t_r} \quad \text{and} \quad \vec{a} = \left. \frac{d^2\vec{w}}{dt^2} \right|_{t_r}$$

Determine $\vec{u} = c \hat{r}_r - \vec{v}$

Fields:

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\hat{r}_r}{(\vec{r}_r \cdot \vec{u})^3} \left[(c^2 - v^2) \vec{u} + \hat{r}_r \times (\vec{u} \times \vec{a}) \right]$$

$$\vec{B} = \frac{1}{c} \hat{r}_r \times \vec{E}$$

2 Fields produced by a charge moving with constant velocity

Consider a charged particle moving with constant velocity. Thus

$$\mathbf{w}(t) = \mathbf{v} t$$

where $\mathbf{v}$ is the velocity.

a) Show that

$$\mathbf{u} = \frac{\mathbf{r} - \mathbf{v} t}{t - t_r}$$

b) Determine an expression for the electric field.

c) Determine an expression for the magnetic field.

Answer: a)
$$\vec{u} = c \hat{r} - \vec{v}$$

$$= \frac{1}{r_r} c \vec{r} - \vec{v}$$

$$\vec{r}_r = \vec{r} - \vec{w}(t_r) = \vec{r} - \vec{v} t_r$$

$$\Rightarrow \vec{u} = \frac{c}{r_r} \vec{r} - \frac{c}{r_r} \vec{v} t_r - \vec{v} \quad r_r = c(t - t_r)$$

$$= \frac{\vec{r} - \vec{v} t_r}{t - t_r} - \vec{v} = \frac{\vec{r} - \vec{v} t_r - \vec{v} t + \vec{v} t_r}{t - t_r}$$

$$= \frac{\vec{r} - \vec{v} t}{t - t_r}$$

b) Here $\vec{a} = 0$. Thus:

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{r_r}{(\vec{r}_r \cdot \vec{u})^3} (c^2 - v^2) \vec{u}$$

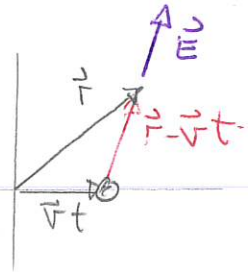
Then $\vec{r}_r \cdot \vec{u} = (c r_r - \vec{v} \cdot \vec{r}_r)$

$$\Rightarrow \vec{E} = \frac{q}{4\pi\epsilon_0} \frac{r_r}{(c r_r - \vec{r}_r \cdot \vec{v})^3} (c^2 - v^2) \frac{\vec{r} - \vec{v} t}{t - t_r}$$

$$\Rightarrow \vec{E} = \frac{q}{4\pi\epsilon_0} \frac{c(t-t_r)(c^2-v^2)(\vec{r}-\vec{v}t)}{(c\hat{r}-\hat{r}\cdot\vec{v})^3 t-t_r}$$

$$\Rightarrow \vec{E} = \frac{q}{4\pi\epsilon_0} \frac{c(c^2-v^2)(\vec{r}-\vec{v}t)\hat{r}}{c^3(\hat{r}-\hat{r}\cdot\vec{v}/c)^3}$$

$$\Rightarrow \vec{E} = \frac{q}{4\pi\epsilon_0} \frac{1}{\hat{r}^2} \frac{1}{(1-\hat{r}\cdot\vec{v}/c)^3} \left(1-\frac{v^2}{c^2}\right)(\vec{r}-\vec{v}t)$$



$$c) \vec{B} = \frac{1}{c} \frac{\vec{r}}{\hat{r}} \times \vec{E}$$

$$\text{Then } \vec{r}_r = \vec{r} - \vec{v}(t_r) = \vec{r} - \vec{v}t_r$$

$$\vec{B} = \frac{1}{c} \frac{q}{4\pi\epsilon_0} \frac{1}{\hat{r}^3} \frac{1}{(1-\hat{r}\cdot\vec{v}/c)^3} \left(1-\frac{v^2}{c^2}\right) \underbrace{(\vec{r}-\vec{v}t_r) \times (\vec{r}-\vec{v}t)}_{\vec{r} \times \vec{v} (t_r-t)}$$

$$\Rightarrow \vec{B} = \frac{1}{c^2} \frac{q}{4\pi\epsilon_0} \frac{1}{\hat{r}^2} \frac{1}{(1-\hat{r}\cdot\vec{v}/c)^3} \left(1-\frac{v^2}{c^2}\right) \vec{v} \times \vec{r}$$

$$c^2 = \frac{1}{\mu_0\epsilon_0}$$

$$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi} \frac{q}{\hat{r}^2} \frac{\vec{v} \times \vec{r}}{(1-\hat{r}\cdot\vec{v}/c)^3} \left(1-\frac{v^2}{c^2}\right)$$

