Thurs: Test II

Covers: Ch ~~8.10~~, 9.1, 9.2, 9.3.1 → 9.3.2.
10.1

HW 9 → 15

Bring: ~~4/22~~ Prev ½ single letter
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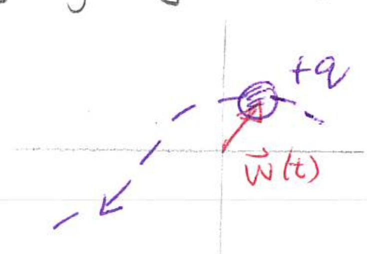
Given: Inside cover eqns.

Prev: 2012 Ex II Q2, 4
2016 Ex II Q1, Q3, Q4a

Potentials from a moving point charge

Consider a particle with charge q following trajectory $\vec{w}(t)$. Then the charge density is:



$$\rho(\vec{r},t) = q\,\delta(\vec{r}-\vec{w}(t))$$

and the current density is

$$\vec{J}(\vec{r},t) = q\,\vec{v}\,\delta(\vec{r}-\vec{w}(t))$$

where

$$\vec{v} = \frac{d\vec{w}}{dt}$$

is the particle velocity

Then we consider computing the scalar potential at time t

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)}{\mathscr{r}}\, d\vec{r}'$$

$$= \frac{q}{4\pi\epsilon_0}\int \frac{1}{|\vec{r}-\vec{r}'|}\, \delta^3(\vec{r}'-\vec{w}(t_r))\, \underbrace{d^3r'}_{d\tau'}$$

where the retarded time is

$$t_r = t - \mathscr{r}/c = t - \frac{|\vec{r}-\vec{r}'|}{c}$$

Thus the delta function becomes:

$$\delta^3\left(\vec{r}' - \vec{w}\left(t - \frac{|\vec{r}-\vec{r}'|}{c}\right)\right)$$

This immediately presents difficulties when evaluating the potential. Note that the three dimensional Dirac-delta function is

$$\delta^3(\vec{r}-\vec{r}_0) = \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$$

Then the one dimensional Dirac-delta function is strictly defined via:

$$\int f(x)\,\underbrace{\delta(x-x_0)}_{\text{must be independent of } x}\,dx = f(x_0)$$

The Dirac-delta function here has form:

$$\delta(x-\underbrace{w_x(t_r)}_{\text{depend on } x,y,z})\,\delta(y-w_y(t_r))\,\delta(z-w_z(t_r))$$

and we cannot use the usual calculation rules.

We will eventually transform integration variables

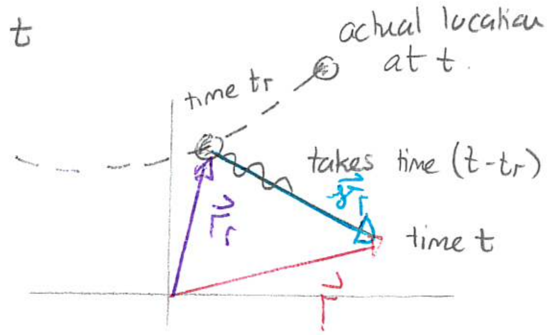
$$\vec{r}' \to \vec{u} = \vec{r}' - \vec{w}(t_r)$$

and this introduces the notion of retarded position:

$$\boxed{\vec{r}_r := \vec{w}(t_r)}$$

The conceptual description of retarded position entails:

1) consider evaluating V at $\vec{r}$ at time t



2) the retarded position is the location of the source at an earlier time (t_r) so that a signal from the source at that point would arrive at r at exactly time t later

We can find the retarded time and retarded position via the retarded separation vector

$$\vec{\mathscr{r}}_r = \vec{r} - \vec{r}_r$$

Then:

$$\mathscr{r}_r = c(t-t_r) \Rightarrow |\vec{r} - \vec{r}_r| = c(t-t_r)$$

$$\Rightarrow |\vec{r} - \vec{w}(t_r)| = c(t-t_r)$$

so we get:

Identify field location $\vec{r}$ and time t. $\to$ Find retarded time by solving $|\vec{r} - \vec{w}(t_r)| = c(t-t_r)$ $\to$ Find retarded position vector $\vec{r}_r = \vec{w}(t_r)$

2. Retardation for a particle traveling with constant velocity

A charged particle travels with constant velocity along the $+x$ axis, passing the origin at $t = 0$. Suppose that one wants to determine the scalar potential at $\mathbf{r} = y\hat{\mathbf{y}}$ at time $t > 0$.

a) Determine an expression for the retarded time.

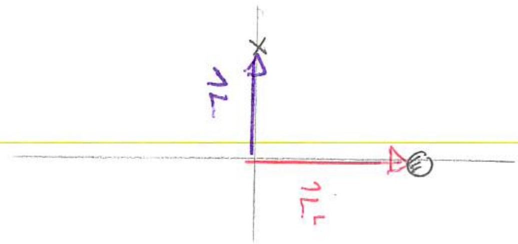
b) Suppose that $v = c/\sqrt{2}$. Determine an expression for the retarded position.

c) Suppose that $v = c/\sqrt{2}$. Determine an expression for the magnitude of the retarded separation vector.

Answer:

a) $\vec{w} = vt\hat{x}$

$\vec{r} = y\hat{y}$



Need

$$|\vec{r} - \vec{w}(t_r)| = c(t - t_r)$$

$$|\vec{r} - v t_r \hat{x}| = c(t-t_r) \Rightarrow |y\hat{y} - vt_r\hat{x}| = c(t-t_r)$$

$$\Rightarrow \sqrt{y^2 + v^2 t_r^2} = ct - t_r$$

$$\Rightarrow y^2 + v^2 t_r^2 = c^2(t-t_r)^2$$

$$\Rightarrow y^2 + v^2 t_r^2 = c^2(t^2 - 2tt_r + t_r^2)$$

$$t_r^2(c^2 - v^2) - 2tc^2 t_r + c^2t^2 - y^2 = 0$$

$$\Rightarrow t_r = \frac{2tc^2 \pm \sqrt{(2tc^2)^2 - 4(c^2t^2 - y^2)(c^2 - v^2)}}{2(c^2 - v^2)}$$

$$t_r = \frac{2tc^2 \pm \sqrt{\cancel{4t^2c^4} - \cancel{4c^4t^2} - 4y^2v^2 + 4y^2c^2 + 4c^2t^2v^2}}{2(c^2-v^2)}$$

$$= \frac{tc^2 \pm \sqrt{c^2v^2t^2 + y^2(c^2-v^2)}}{(c^2-v^2)}$$

$$= \frac{tc^2 \pm c^2\sqrt{\frac{v^2t^2}{c^2} + \frac{y^2}{c^2}(1-v^2/c^2)}}{c^2-v^2}$$

$$= \frac{1}{1-v^2/c^2}\left[t \pm \sqrt{\frac{v^2t^2}{c^2} + (1-v^2/c^2)\frac{y^2}{c^2}}\right]$$

The + sign would give that the entire quantity is $> t$. This is not possible. So

$$t_r = \frac{1}{1-v^2/c^2}\left[t - \sqrt{\frac{v^2t^2}{c^2} + (1-v^2/c^2)\frac{y^2}{c^2}}\right]$$

b) Here

$$t_r = \frac{1}{1-1/2}\left[t - \sqrt{\frac{1}{2}t^2 + \left(1-\frac{1}{2}\right)\frac{y^2}{c^2}}\right]$$

$$= 2\left[t - \frac{1}{\sqrt{2}}\sqrt{t^2 + y^2/c^2}\right] = 2t\left[1 - \frac{1}{\sqrt{2}}\sqrt{1 + y^2/c^2t^2}\right]$$

$$\vec{r}_r = \vec{w}(t_r) = v t_r \hat{x}$$

$$= 2v\left[t - \frac{1}{\sqrt{2}}\sqrt{t^2 + y^2/c^2}\right]\hat{x}$$

$$= 2vt\left[1 - \frac{1}{\sqrt{2}}\sqrt{1 + \frac{y^2}{c^2t^2}}\right]\hat{x}$$

c) $\vec{\mathscr{r}}_r = |\vec{r} - \vec{r}_r|$

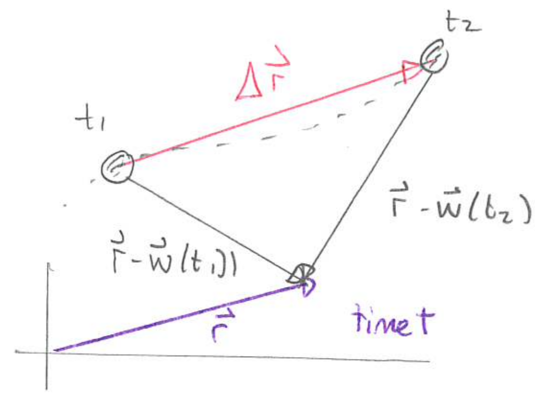
$$\Rightarrow \mathscr{r}_r = \sqrt{y^2 + v^2 t_r^2}$$

$$\mathscr{r}_r^{\,2} = y^2 + v^2 t_r^2$$

$$= y^2 + v^2 4 t^2 \left[1 - \frac{1}{\sqrt{2}} \sqrt{1 + y^2/c^2 t^2} \right]^2$$

We see how it is possible to find such a retarded position and retarded time. We need to check that there is only one possibility.

Suppose that there are two, denoted t_1 and t_2. Then:



$$|\vec{r}-\vec{w}(t_1)| = c(t-t_1)$$

$$|\vec{r}-\vec{w}(t_2)| = c(t-t_2)$$

$$\Rightarrow \quad |\vec{r}-\vec{w}(t_1)| - |\vec{r}-\vec{w}(t_2)| = c(t_2-t_1)$$

Now consider the diagram.

$$\Delta\vec{r} = \vec{w}(t_2) - \vec{w}(t_1)$$

$$= (\vec{r}-\vec{w}(t_1)) - (\vec{r}-\vec{w}(t_2))$$

$$\Rightarrow (\vec{r}-\vec{w}(t_1)) = \Delta\vec{r} + (\vec{r}-\vec{w}(t_2))$$

By the triangle inequality

$$|\vec{r}-\vec{w}(t_1)| \leq \Delta r + |\vec{r}-\vec{w}(t_2)|$$

$$\Rightarrow \quad \Delta r \geq |\vec{r}-w(t_1)| - |\vec{r}-w(t_2)| = c(t_2-t_1)$$

So the particle would have to move faster than the speed of light. We get.

$$\boxed{\text{For any particle trajectory, there can be at most one retarded position that contributes to the potential at any single field point at any single time.}}$$

Retarded position and the potential integral

Note that the retarded position has an important mathematical interpretation in the context of

$$\vec{u} := \underbrace{\vec{r}\,' - \vec{w}(t_r)}_{\text{function of } \vec{r}\,'}$$

The l.h.s can be regarded as three functions of x', y', z'. So

$$u_x = x' - w_x\left(t - \frac{1}{c}\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\right) = u_x(x', y', z')$$

given

observation choice

$$u_y = y' - w_y(\ \ldots\ldots\ldots\ \ ) = u_y(x', y', z')$$

$$u_z = z' - w_z(\ \ldots\ldots\ldots\ \ ) = u_z(x', y', z')$$

We can write this compactly in the form

$$\vec{u} = \vec{u}(\vec{r}\,')$$

Then the retarded position $\vec{r}_r$ is the value of $\vec{r}\,'$ such that

$$0 = \vec{u}(\vec{r}\,') \Rightarrow 0 = \vec{u}(\vec{r}_r) \Rightarrow \underbrace{\vec{r}_r = \vec{w}(t_r)}_{\text{does not refer to primed}}$$

This is essential in the transformation of the integral.

The potential integral has the form:

$$\int f(\vec{r}')\, \delta^3(\vec{r}' - \vec{w}(t_r))\, d^3r'$$

$$= \int f(x',y',z')\, \delta(x' - w_x(t_r))\, \delta(y' - w_y(t_r))\, \delta(z' - w_z(t_r))\, dx'dy'dz'$$

Then the relevant transformation is:

$$\vec{u} = \vec{r}' - \vec{w}(t_r)$$

and this has an inverse transformation (complicated)

$$\vec{r}' = \vec{r}'(\vec{u}) \quad \Rightarrow \quad \begin{aligned} x' &= x'(u_x, u_y, u_z) \\ y' &= y'(u_x, u_y, u_z) \\ z' &= z'(u_x, u_y, u_z) \end{aligned}$$

So, in the integral,

$$f(x',y',z') = f(\vec{r}') \rightarrow f(\vec{r}'(\vec{u}))$$

$$\delta^3(\vec{r}' - \vec{w}(t_r)) \longrightarrow \delta^3(\vec{u})$$

and

$$dx'dy'dz' \rightarrow \frac{1}{J}\, du_x\, du_y\, du_z$$

where the Jacobian is:

$$J = \begin{vmatrix} \frac{\partial u_x}{\partial x'} & \frac{\partial u_y}{\partial x'} & \frac{\partial u_z}{\partial x'} \\ \frac{\partial u_x}{\partial y'} & \frac{\partial u_y}{\partial y'} & \frac{\partial u_z}{\partial y'} \\ \frac{\partial u_x}{\partial z'} & \frac{\partial u_y}{\partial z'} & \frac{\partial u_z}{\partial z'} \end{vmatrix} \equiv J(\vec{r}'(\vec{u}))$$

Thus

$$\int f(\vec{r}')\, \delta^3(\vec{r}' - \vec{w}(t_r))\, d^3r' = \int f(\vec{r}'(\vec{u}))\, \frac{1}{J(\vec{r}'(\vec{u}))}\, \delta^3(\vec{u})\, d^3u$$

$$= \frac{f(\vec{r}'(\vec{0}))}{J(\vec{r}'(\vec{0}))}$$

This means:

1) express f as a function of $\vec{r}'$ and evaluate at $\vec{r}'$ s.t. $\vec{r}'$ gives $\vec{u}=0$. This is exactly $\vec{r}_r$.

2) express J as a function of $\vec{r}'$ and evaluate at $\vec{r}'$ s.t. this gives $\vec{u}=0$. This is also exactly $\vec{r}_r$.

In the case of the potential

$$f(\vec{r}') = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r}-\vec{r}'|} \Rightarrow f(\vec{r}'(\vec{0})) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r}-\vec{r}_t|}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{\mathscr{r}_r}$$

where $\vec{\mathscr{r}}_r = \vec{r} - \vec{r}_r$. Thus we have reached:

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{\mathscr{r}_r} \frac{1}{J(\vec{r}'(\vec{0}))}$$

It remains to calculate the Jacobian.

<u>Theorem:</u> For the situation described above

$$J = 1 - \frac{\vec{v} \cdot \vec{\mathscr{r}}_r}{c\, \mathscr{r}_r}$$

where $\vec{\mathscr{r}}_r = \vec{r} - \vec{r}_r$. $\boxtimes$

<u>Proof:</u>

$$u_x = x' - w_x(t_r)$$

$$u_y = y' - w_y(t_r)$$

$$u_z = z' - w_z(t_r)$$

$$J = \begin{vmatrix} \frac{\partial u_x}{\partial x'} & \frac{\partial u_y}{\partial x'} & \frac{\partial u_z}{\partial x'} \\ \frac{\partial u_x}{\partial y'} & \frac{\partial u_y}{\partial y'} & \frac{\partial u_z}{\partial y'} \\ \frac{\partial u_x}{\partial z'} & \frac{\partial u_y}{\partial z'} & \frac{\partial u_z}{\partial z'} \end{vmatrix} = \begin{vmatrix} 1 - \overbrace{\frac{dw_x}{dt}}^{v_x} \frac{\partial t_r}{\partial x'} & -\overbrace{\frac{dw_y}{dt}}^{v_y} \frac{\partial t_r}{\partial x'} & -\overbrace{\frac{dw_z}{dt}}^{v_z} \frac{\partial t_r}{\partial x'} \\ -\frac{dw_x}{dt} \frac{\partial t_r}{\partial y'} & 1 - \frac{dw_y}{dt} \frac{\partial t_r}{\partial y'} & -\frac{dw_z}{dt} \frac{\partial t_r}{\partial y'} \\ -\frac{dw_x}{\partial t} \frac{\partial t_r}{\partial z'} & -\frac{dw_y}{dt} \frac{\partial t_r}{\partial z'} & 1 - \frac{dw_z}{dt} \frac{\partial t_r}{\partial z'} \end{vmatrix}$$

$$= \left(1 - v_x \frac{\partial t_r}{\partial x'}\right) \left[\left(1 - v_y \frac{\partial t_r}{\partial y'}\right)\left(1 - v_z \frac{\partial t_r}{\partial z'}\right) - v_y v_z \frac{\partial t_r}{\partial z'} \frac{\partial t_r}{\partial y'} \right]$$

$$+ v_y \frac{\partial t_r}{\partial x'} \left[-v_x \frac{\partial t_r}{\partial y'} \left(1 - v_z \frac{\partial t_r}{\partial z'}\right) - v_x v_z \frac{\partial t_r}{\partial z'} \frac{\partial t_r}{\partial x'} \right]$$

$$- v_z \frac{\partial t_r}{\partial x'} \left[v_x v_y \frac{\partial t_r}{\partial y'} \frac{\partial t_r}{\partial z'} + v_x \frac{\partial t_r}{\partial z'} \left(1 - v_y \frac{\partial t_r}{\partial y'}\right) \right.$$

$$\Rightarrow \quad J = \left(1 - v_x \frac{\partial t_r}{\partial x'}\right)\left[1 - v_y \frac{\partial t_r}{\partial y'} - v_z \frac{\partial t_r}{\partial z'}\right]$$

$$- v_y v_x \frac{\partial t_r}{\partial x'}\frac{\partial t_r}{\partial y'} - v_x v_z \frac{\partial t_r}{\partial x'}\frac{\partial t_r}{\partial z'}$$

$$= 1 - v_x \frac{\partial t_r}{\partial x'} - v_y \frac{\partial t_r}{\partial y'} - v_z \frac{\partial t_r}{\partial z'}$$

Now $\quad t_r = t - \dfrac{|\vec{r} - \vec{r}\,'|}{c}$

$$= t - \frac{1}{c}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}$$

$$\Rightarrow \quad \frac{\partial t_r}{\partial x'} = -\frac{1}{c}\left(\frac{1}{\cancel{2}}\right)\frac{\cancel{2}(-1)(x-x')}{[\;.\;.\;.\;.\;]^{1/2}} = \frac{x-x'}{c[\;.\;.\;.\;]}$$

$$\Rightarrow \quad \vec{J} = 1 - \frac{1}{c}\frac{\vec{v}\cdot\vec{\mathscr{r}}}{\mathscr{r}} \qquad \text{where} \qquad \vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$$

This must be evaluated when $\vec{r}\,' = \vec{r}_r$. Then $\vec{\mathscr{r}} \to \vec{\mathscr{r}}_r$ and that gives the result.

Substitution gives

$$V(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}_r}\frac{1}{1-\vec{v}\cdot\vec{\mathscr{r}}_r/c\mathscr{r}_r}$$

$$V(\vec{r},t) = \frac{qc}{4\pi\epsilon_0}\frac{1}{c\mathscr{r}_r - \vec{v}\cdot\vec{\mathscr{r}}_r}$$

We can apply the same reasoning + type of derivation to the vector potential:

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau'$$

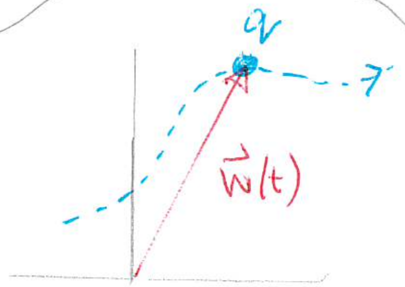
This will give

$$\vec{A} = \frac{\mu_0}{4\pi}qc\frac{\vec{v}}{c\mathscr{r}_r - \vec{v}\cdot\vec{\mathscr{r}}_r}$$

Clearly in this case

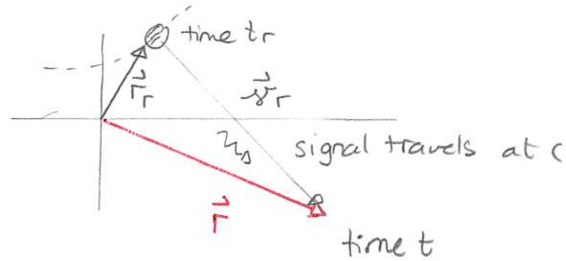
$$\vec{A} = \mu_0\epsilon_0\vec{v}(t_r)V(\vec{r},t) = \frac{1}{c^2}\vec{v}(t_r)\,V(\vec{r},t)$$

This now gives a method for determining the potentials due to a moving point charge.



Specify trajectory $\vec{w}(t)$

Want potential at location $\vec{r}$ at time t.



Find retarded time t_r by solving

$$c(t-t_r) = |\vec{r} - \vec{w}(t_r)|$$

Calculated retarded position

$$\vec{r}_r = \vec{w}(t_r)$$

and retarded separation

$$\vec{\mathscr{r}}_r = \vec{r} - \vec{r}_r$$

Scalar potential

$$V(\vec{r},t) = \frac{qc}{4\pi\epsilon_0} \frac{1}{\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r}$$

Vector potential

$$\vec{A}(\vec{r},t) = \frac{\mu_0 q c}{4\pi} \frac{\vec{v}(t_r)}{\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r}$$

Fields

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

These are called the Liénard-Wiechert potentials