Phys 312 2021

Lecture 14

Tues: HW by 5pm

Thurs: Read 9.3.2, 9.3.3.

Maxwell's equations in a vacuum

Maxwell's equations, in general are:

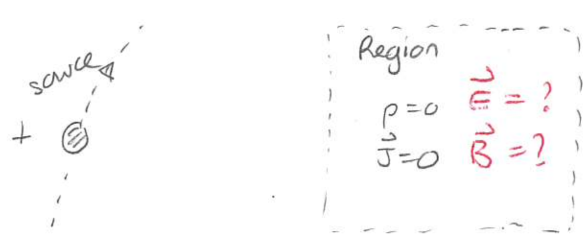
$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

The strategy for managing these is to specify source charge density ρ and source current density $\vec{J}$ and find the electric and magnetic fields that will satisfy the equations. The simplest case is that where $\rho = 0$ and $\vec{J} = 0$. If these are zero in all space then the fields are also zero. However, if they are zero in some region of space with charges and currents beyond that region then the fields could be non-zero.



In such a region,

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

We use a general vector identity

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

and this gives:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

A similar derivation gives

$$\nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

Thus, in a region where there are no source charges or currents (typically a vacuum)

$$\boxed{\begin{aligned}\nabla^2\vec{E} &= \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}\\ \nabla^2\vec{B} &= \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}\end{aligned}}$$

Each of these is a set of three wave equations:

$$\nabla^2 E_x = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \qquad \nabla^2 B_x = \mu_0 \epsilon_0 \frac{\partial^2 B_x}{\partial t^2}$$

$$\nabla^2 E_y = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

$$\vdots \qquad\qquad \vdots$$

These all have the form

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

where the speed of the waves is given via

$$\frac{1}{v^2} = \mu_0 \epsilon_0 \quad \Rightarrow \quad v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Substituting known values for these constants results in a wavespeed close to the measured speed of light. Thus we find

> In a vacuum region (no source currents or charges), Maxwell's equations imply that waves of electric and magnetic fields exist and that these travel with speed of light
>
> $$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Demo: 1) PSU-S

2) PhET radio waves

These are electromagnetic waves.

Sinusoidal Traveling Waves

The simplest mathematical forms of waves are sinusoidal waves. We expect solutions of the form

$$\vec{E} = E_0 \sin(\vec{k}\cdot\vec{r} - \omega t)$$

where $\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$ is a wavenumber vector and ω an angular frequency. It is more convenient to use the complex exponential form:

$$\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \longrightarrow \vec{E} = Re[\tilde{\vec{E}}]$$

Real solutions

$$\tilde{\vec{B}} = \tilde{\vec{B}}_0 e^{i(\vec{k}'\cdot\vec{r} - \omega' t)} \longrightarrow \vec{B} = Re[\tilde{\vec{B}}]$$

Note that all directions of travel are account for via all directions of $\vec{k}$. We investigate such solutions. Questions are:

1) Are there independent solutions for electric and magnetic fields?

2) How are the directions of the fields related to the direction of propagation?

3) How are the directions of the $\vec{B}$ fields related to the directions of the $\vec{E}$ fields?

1 Electromagnetic sinusoidal plane waves

Consider

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}.$$

a) Using the divergence of the electric field, determine constraints on the direction of $\tilde{\mathbf{E}}_0$.

b) Using the curl of the electric field, show that the magnetic field wave propagates in the same direction as the electric field and with the same wavenumber and frequency.

c) If $\tilde{\mathbf{E}}_0 = \tilde{E}_0\hat{\mathbf{x}}$ use the curl of the electric field to determine the direction and magnitude of the magnetic field.

Answer: a) $\vec{\nabla}\cdot\vec{\tilde{E}} = 0$

$$\Rightarrow \frac{\partial \tilde{E}_x}{\partial x} + \frac{\partial \tilde{E}_y}{\partial y} + \frac{\partial \tilde{E}_z}{\partial z} = 0$$

$$\tilde{E}_x = \tilde{E}_{ox} e^{i(kz-\omega t)} \Rightarrow \frac{\partial \tilde{E}_x}{\partial x} = 0$$

$$\tilde{E}_y = \tilde{E}_{oy} e^{i(kz-\omega t)} \Rightarrow \frac{\partial \tilde{E}_y}{\partial y} = 0$$

$$\tilde{E}_z = \tilde{E}_{oz} e^{i(kz-\omega t)} \Rightarrow \frac{\partial \tilde{E}_z}{\partial z} = ik\tilde{E}_{oz}$$

Thus $\vec{\nabla}\cdot\vec{\tilde{E}} = 0 \Rightarrow E_{oz} = 0$

$\Rightarrow \vec{E}$ is in the xy plane

$\Rightarrow \vec{E}$ is perpendicular to $\hat{z}$

$\Rightarrow \vec{E}$ " " " direction of propagation

b) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \tilde{E}_{ox} e^{i(..)} & \tilde{E}_{oy} e^{i(...)} & \tilde{E}_{oz} e^{i(...)} \end{vmatrix} = -\frac{\partial B_x}{\partial t}\hat{x} - \frac{\partial B_y}{\partial t}\hat{y} - \frac{\partial B_z}{\partial t}\hat{z}$$

Then this gives

$$\hat{x}\left[-\frac{\partial}{\partial z}\tilde{E}_{oy}e^{i(kz-\omega t)}\right]+\hat{y}\left[\frac{\partial}{\partial z}\tilde{E}_{ox}e^{i(kz-\omega t)}\right]=-\frac{\partial B_x}{\partial t}\hat{x}-\frac{\partial B_y}{\partial t}\hat{y}-\frac{\partial B_z}{\partial t}\hat{z}$$

$$\Rightarrow\quad -ik\tilde{E}_{oy}e^{i(kz-\omega t)}\,\hat{x}+ik\tilde{E}_{ox}e^{i(kz-\omega t)}\,\hat{y}=\dots$$

$$\Rightarrow\begin{cases}\dfrac{\partial B_x}{\partial t}=+ik\tilde{E}_{oy}e^{i(kz-\omega t)}\\[2ex]\dfrac{\partial B_y}{\partial t}=-ik\tilde{E}_{ox}e^{i(kz-\omega t)}\\[2ex]\dfrac{\partial B_z}{\partial t}=0\end{cases}\quad\Rightarrow\begin{cases}B_x=-\dfrac{k}{\omega}\tilde{E}_{oy}e^{i(kz-\omega t)}\\[2ex]B_y=+\dfrac{k}{\omega}\tilde{E}_{ox}e^{i(kz-\omega t)}\\[2ex]\underbrace{B_z=0}_{\vec{\nabla}\cdot\vec{B}=0\text{ would give this.}}\end{cases}$$

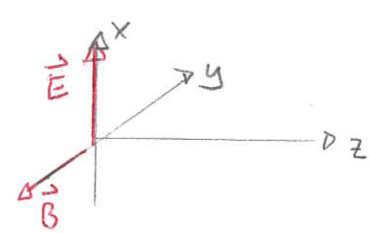
Now $k/\omega=\frac{1}{c}$ gives:

$$\tilde{\vec{B}}=-\frac{1}{c}\left[\tilde{E}_{oy}e^{i(kz-\omega t)}\,\hat{x}-\tilde{E}_{ox}e^{i(kz-\omega t)}\,\hat{y}\right]$$

This is again a wave:

* propagating along $+z$
* same k as $\vec{E}$
* same ω as $\vec{E}$

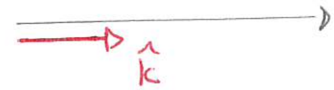
c) Here $\tilde{E}_{ox}=\tilde{E}_0$, $\tilde{E}_{oy}=0$ $\quad\Rightarrow\quad \tilde{\vec{B}}=-\frac{1}{c}\tilde{E}_0e^{i(kz-\omega t)}\,\hat{y}$



Properties of sinusoidal electromagnetic waves

The previous example illustrated general properties of sinusoidal electromagnetic waves. Consider a wave with wavenumber vector $\vec{k}$. This propagates along $\hat{k}$ and has angular frequency

$$\omega = ck$$



Then the complex representation of this is:

$$\tilde{\vec{E}} = \tilde{\vec{E}}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

where

$$\tilde{\vec{E}}_0 = E_{0x} e^{i\delta_x} \hat{x} + E_{0y} e^{i\delta_y} \hat{y} + E_{0z} e^{i\delta_z} \hat{z}$$

From this we can deduce:

1) The electric field is perpendicular to the direction of propagation:

$$\tilde{\vec{E}} \cdot \vec{k} = 0$$

2) The magnetic field has the same wavenumber vector and frequency

$$\vec{B} = \tilde{\vec{B}}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

3) The magnetic field is perpendicular to the direction of propagation

$$\tilde{\vec{B}} \cdot \vec{k} = 0$$

4) The magnetic field and electric field are perpendicular and

$$\vec{B} = \frac{1}{\omega}(\vec{k} \times \vec{E}) \Leftrightarrow \vec{k} = \omega(\vec{E} \times \vec{B})/E^2$$

5) The magnitudes of the fields are related by $B = \frac{1}{c} E$

The proofs are:

1) Direction of $\vec{E}$

$$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{\nabla}\cdot\left[\tilde{\vec{E}}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$

$$\Rightarrow \underbrace{\cancel{\left(\vec{\nabla}\cdot\tilde{\vec{E}}_0\right)}}_{0}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \tilde{\vec{E}}_0\cdot\left[\vec{\nabla}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$

constant

Then $\vec{\nabla}\left[e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = i\vec{k}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$$\Rightarrow i\,\tilde{\vec{E}}_0\cdot\vec{k}\,\cancel{e^{i(\vec{k}\cdot\vec{r}-\omega t)}} = 0 \Rightarrow \tilde{\vec{E}}_0\cdot\vec{k} = 0$$

$$\Rightarrow \vec{E}\cdot\vec{k} = 0$$

2) Magnetic field

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \vec{\nabla}\times\left[\tilde{\vec{E}}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = -\frac{\partial\vec{B}}{\partial t}$$

$$\Rightarrow e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,\underbrace{\cancel{\vec{\nabla}\times\tilde{\vec{E}}_0}}_{0} - \tilde{\vec{E}}_0\times\vec{\nabla}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} = -\frac{\partial\vec{B}}{\partial t}$$

constant

$$\Rightarrow \frac{\partial\vec{B}}{\partial t} = -\tilde{\vec{E}}_0 \times i\vec{k}\; e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= i\left(\vec{k}\times\tilde{\vec{E}}_0\right)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Integration gives $\vec{B} = \frac{1}{\omega}\left(\vec{k}\times\tilde{\vec{E}}_0\right) e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

We have the same wavenumber, the same frequency and

$$\tilde{\vec{B}} = \tilde{\vec{B}}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

where $\tilde{\vec{B}}_0 = \frac{1}{\omega}\left(\vec{k}\times\tilde{\vec{E}}_0\right)$

3) Direction of magnetic field

Using $\vec\nabla\cdot\vec B = 0$ and the same reasoning as for the electric field gives the result that $\vec B\cdot\vec k = 0$.

4) Magnetic and electric field

From $\quad \tilde{\vec B}_0 = \frac{1}{\omega}\,\vec k\times\tilde{\vec E}_0 \quad$ we get

$$\vec B = \frac{1}{\omega}(\vec k\times\vec E)$$

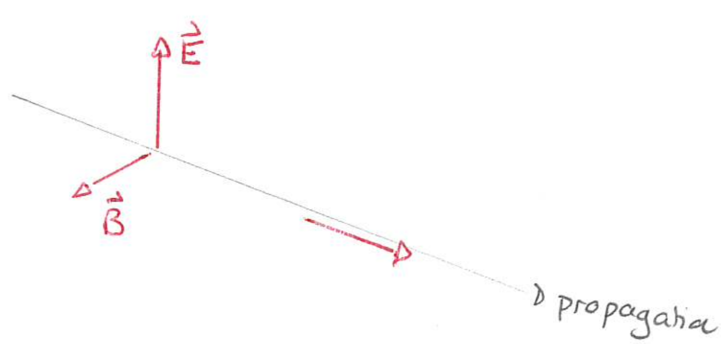
Then
$$\vec E\times\vec B = \frac{1}{\omega}\,\vec E\times(\vec k\times\vec E)$$
$$= \frac{1}{\omega}\,\vec k\,(\vec E\cdot\vec E) - \vec E\,\cancel{(\vec k\cdot\vec E)}\ 0$$
$$\Rightarrow \quad \vec k = \omega\,(\vec E\times\vec B)/E^2$$

5) Magnitudes

$$B = \frac{1}{\omega}\,kE\sin 90^\circ \quad\Rightarrow\quad B = \frac{k}{\omega}E \Rightarrow B = \frac{1}{c}E$$

Thus we have:

The electromagnetic wave is completely specified by $\vec k$ and $\tilde{\vec E}_0$. These determine the electric and magnetic fields completely

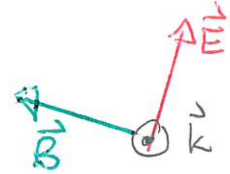


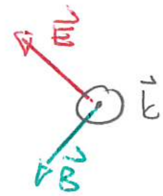
These statements will be true for any plane wave solution:

$$\vec E(\vec r,t) = \tilde{\vec E}_0\,g(\vec k\cdot\vec r-\omega t)$$

Demo: PSU-S waves.

Polarization of electromagnetic waves





For any given wavenumber vector $\vec{k}$ there are many possible directions for the electric field. We can focus on a single location and consider these vectors at this location. Then at say $\vec{r}=0$ we get.

$$\tilde{\vec{E}} = \tilde{\vec{E}}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)} = \tilde{\vec{E}}_0 \, e^{-i\omega t}$$

If the direction of propagation is long z then

$$\tilde{\vec{E}}_0 = E_{ox} e^{i\delta_x} \hat{x} + E_{oy} e^{i\delta_y} \hat{y}$$

$$\Rightarrow \tilde{\vec{E}} = E_{ox} e^{i(-\omega t + \delta_x)} \hat{x} + E_{oy} e^{i(-\omega t + \delta_y)} \hat{y}$$

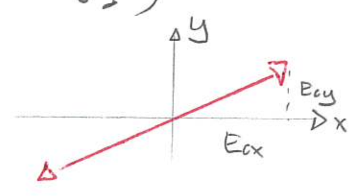
Taking the real part gives:

$$\vec{E} = E_{ox} \cos(-\omega t + \delta_x) \hat{x} + E_{oy} \cos(-\omega t + \delta_y) \hat{y}$$

$$\Rightarrow \boxed{\vec{E} = E_{ox} \cos(\omega t - \delta_x) \hat{x} + E_{oy} \cos(\omega t - \delta_y) \hat{y}}$$

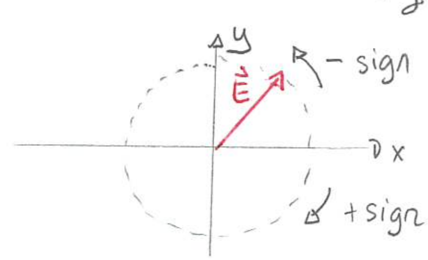
This then describes various states of polarization:

1) <u>linear polarization</u> $\delta_x = \delta_y = \delta \Rightarrow \vec{E} = \cos(\omega t - \delta)(E_{ox}\hat{x} + E_{oy}\hat{y})$



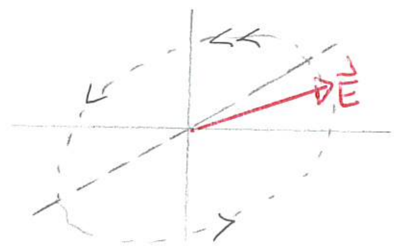
2) <u>circular polarization</u> $\delta_y = \delta_x \pm \pi/2 \Rightarrow \vec{E} = E_0 \cos(\omega t - \delta_x)\hat{x} \mp E_0 \sin(\omega t - \delta_x)\hat{y}$

$E_{ox} = E_{oy} = E_0$



3) <u>elliptical polarization</u>

This is the general case for all $\delta_x, \delta_y, E_{0x}, E_{0y}$ that are independent of time.



For pure sinusoidal solutions these are the only possibilities. If $\tilde{\vec{E}}_0$ is allowed to vary randomly with time then there are other partially polarized possibilities but these are not pure sinusoidal solutions.

Energy and electromagnetic waves

Recall that there is an energy associated with electromagnetic fields and also a rate of transport of energy. Within any closed region the total energy is

$$\underbrace{K}_{\text{kinetic of particles}} + \underbrace{U}_{\text{energy in fields}}$$

Then

$$\frac{d}{dt}(K+U) = -\oint_S \vec{S}\cdot d\vec{a}$$

where the Poynting vector is:

$$\boxed{\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}}$$

and the energy stored in the fields is

$$U = \frac{\epsilon_0}{2}\int\left(\vec{E}\cdot\vec{E} + \frac{1}{\epsilon_0\mu_0}\vec{B}\cdot\vec{B}\right)d\tau \Rightarrow \boxed{U = \frac{\epsilon_0}{2}\int_R\left(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B}\right)d\tau}$$

The Poynting vector quantifies the rate at which energy flows through any surface. We interpret:

1) direction of $\vec{S}$ $\equiv$ direction of energy flow

2) magnitude of $\vec{S}$ $\equiv$ energy flow per second per unit area.

2 Energy for electromagnetic plane waves

Consider plane waves that propagate along the $+z$ direction.

a) Show that the energy density is

$$u = \epsilon_0 \mathbf{E} \cdot \mathbf{E}.$$

b) Show that the Poynting vector is

$$\mathbf{S} = \epsilon_0 c E^2 \, \hat{\mathbf{z}} = uc \, \hat{\mathbf{z}}.$$

Answer: a) The energy density is

$$\frac{\epsilon_0}{2} \vec{E}\cdot\vec{E} + \frac{1}{2\mu_0} \vec{B}\cdot\vec{B} = \frac{\epsilon_0}{2}\left[\vec{E}\cdot\vec{E} + \frac{1}{\mu_0\epsilon_0}\vec{B}\cdot\vec{B}\right]$$

$$= \frac{\epsilon_0}{2}\left[\vec{E}\cdot\vec{E} + c^2 \vec{B}\cdot\vec{B}\right]$$

Then $\vec{E}\cdot\vec{E} = E^2$; $\vec{B}\cdot\vec{B} = B^2 = \frac{1}{c^2}E^2 = \frac{1}{c^2}\vec{E}\cdot\vec{E}$. Thus the energy density is

$$u = \epsilon_0 \vec{E}\cdot\vec{E}$$

b) $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$. In general $\vec{B} = \frac{1}{\omega}\vec{k}\times\vec{E}$

$$\vec{S} = \frac{1}{\mu_0\omega}\vec{E}\times(\vec{k}\times\vec{E}) = \frac{1}{\mu_0\omega}\left[\vec{k}\,(\vec{E}\cdot\vec{E}) - \vec{E}\,\underbrace{\cancel{(\vec{k}\cdot\vec{E})}}_{\text{for waves}}\right]$$

$$= \frac{1}{\mu_0\omega}\vec{k}\left(\vec{E}\cdot\vec{E}\right)$$

$$= \frac{k}{\mu_0\omega}(\vec{E}\cdot\vec{E})\,\hat{z} = \frac{1}{\mu_0 c}(\vec{E}\cdot\vec{E})\,\hat{z}$$

$$= \frac{\epsilon_0}{\epsilon_0\mu_0 c}\vec{E}\cdot\vec{E}\,\hat{z}$$

$$= \epsilon_0 c\,\vec{E}\cdot\vec{E}\,\hat{z}$$

$$= uc\,\hat{z}$$