Phys 312 2021

Lecture 13

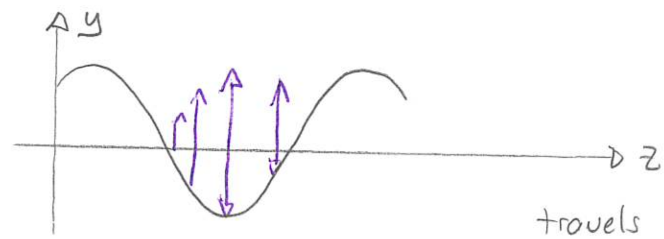
Thurs: Seminar

Fri: HW

Tues: Read 9.3.1 → 9.3.2

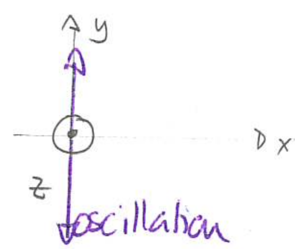
Polarization

So far the types of waves that we have considered all propagate along the z direction while the oscillations are all along a single perpendicular direction. Such waves can always be represented by a displacement which is a scalar:

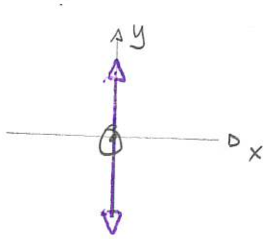


$$f(z,t) = A\cos(kz-\omega t) \equiv y$$

viewed along direction of approach

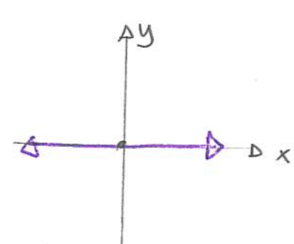


Now the medium could oscillate along any direction in the xy plane and we need to describe this range of possibilities. Some examples are:



$$\vec{f}(z,t) = A\hat{y}\cos(kz-\omega t)$$

vertical



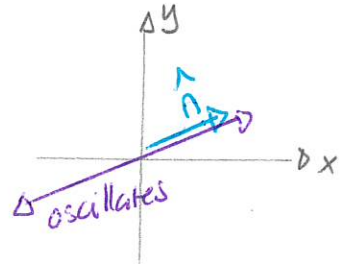
$$\vec{f}(z,t) = A\hat{x}\cos(kz-\omega t)$$

horizontal

To capture all such possibilities the oscillating quantity must be described by a vector $\vec{f}$. The information about the direction of this vector will be described in terms of the polarization of the wave

Polarization ~ information about direction of oscillation.

An important category of such waves are those for which the oscillations are all along a single direction that is perpendicular to the direction of travel of the wave. If we denote this direction by $\hat{n}$ then the wave is described via



$$\vec{f}(z,t) = A\hat{n}\cos(kz-\omega t)$$

or

$$\tilde{\vec{f}}(z,t) = \tilde{A}\hat{n}\, e^{i(kz-\omega t)}$$

where $\tilde{A}$ is complex, i.e. $\tilde{A} = Ae^{i\delta}$. These are linearly polarized waves. Then $\hat{n}$ describes the line of polarization.

These are all transverse waves in the sense that the direction of propagation (z) is perpendicular to the direction of oscillation. The most general complex exponential transverse wave that travels right is

$$\tilde{\vec{f}}(z,t) = \tilde{\vec{A}}\, e^{i(kz-\omega t)}$$

where

$$\tilde{\vec{A}} = \tilde{A}_x\hat{x} + \hat{A}_y\hat{y}$$

and

$$\tilde{A}_x = A_x e^{i\delta_x} \qquad , \qquad \tilde{A}_y = A_y e^{i\delta_y}$$

For such a wave:

$$\tilde{\vec{f}} = A_x e^{i(kz-\omega t+\delta_x)}\,\hat{x} + A_y e^{i(kz-\omega t+\delta_y)}\,\hat{y}$$

results in a real wave

$$\vec{f} = \mathrm{Re}[\tilde{\vec{f}}]$$

$$\vec{f} = A_x \cos(kz-\omega t+\delta_x)\,\hat{x} + A_y \cos(kz-\omega t+\delta_y)\,\hat{y}$$

The behavior of this wave depends on $A_x, A_y, \delta_x, \delta_y$ and is most conveniently analyzed by considering the oscillations at one location as time passes.

1 Polarization of waves

The complex representation of "vector" sinusoidal waves is:

$$\tilde{\mathbf{f}} = \tilde{\mathbf{A}} e^{i(kz-\omega t)}$$

where $\tilde{\mathbf{A}}$ is a complex amplitude vector.

a) Specify the complex amplitude vector for waves polarized along the x axis.

b) Specify the complex amplitude vector for waves polarized along the axis angled at 45° between the x and y axes.

In general

$$\tilde{\mathbf{A}} = \tilde{A}_x \hat{\mathbf{x}} + \tilde{A}_y \hat{\mathbf{y}}$$

where $\tilde{A}_x = A_x e^{i\delta_x}$ and $\tilde{A}_y = A_y e^{i\delta_y}$ with A_x, A_y and δ_x, δ_y all real.

c) Suppose that $\tilde{A}_x = Ae^{-i\pi/2}$ and $\tilde{A}_y = A$ with $A > 0$. Determine a real expression for this wave. Sketch the vector describing the wave at $z = 0$ as time passes.

Answer: a) $\vec{A}$ is along $\hat{x}$. So $\vec{A} = A\hat{x}$. A general complex form would be

$$\vec{A} = Ae^{i\delta_x}\hat{x}$$

$$\Rightarrow \tilde{\vec{f}} = Ae^{i\delta_x}\hat{x}\, e^{i(kz-\omega t)} = A\hat{x}\, e^{i(kz-\omega t+\delta_x)}$$

Note the real wave is

$$\vec{f} = A\cos(kz-\omega t+\delta_x)\,\hat{x}$$

b) $\vec{A}$ is along $\hat{n} = \frac{1}{\sqrt{2}}(\hat{x}+\hat{y})$

$$\tilde{\vec{A}} = A\frac{1}{\sqrt{2}}(\hat{x}+\hat{y})e^{i\delta}$$

$$\tilde{\vec{f}} = A\frac{1}{\sqrt{2}}(\hat{x}+\hat{y})e^{i\delta}e^{i(kz-\omega t)}$$

$$\vec{f} = A\frac{1}{\sqrt{2}}(\hat{x}+\hat{y})\cos(kz-\omega t+\delta)$$

c) $$\tilde{\vec{f}} = \tilde{\vec{A}} e^{i(kz-\omega t)}$$

$$= \left(\tilde{A}_x \hat{x} + \tilde{A}_y \hat{y}\right) e^{i(kz-\omega t)}$$

$$= \left(A e^{-i\pi/2} \hat{x} + A\hat{y}\right) e^{i(kz-\omega t)}$$

$$= A e^{i(kz-\omega t-\pi/2)} \hat{x} + A e^{i(kz-\omega t)} \hat{y}$$

The real part is

$$\vec{f} = A\cos(kz-\omega t-\pi/2)\,\hat{x} + A\cos(kz-\omega t)\,\hat{y}$$

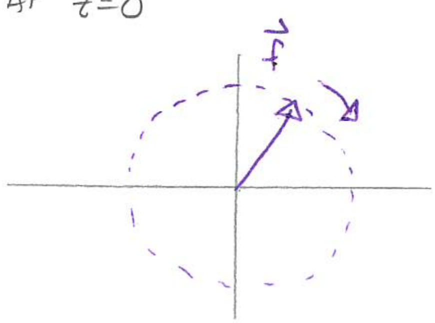
At $z=0$

$$\vec{f} = A\cos(\omega t+\pi/2)\,\hat{x} + A\cos(\omega t)\,\hat{y}$$

$$= -A\sin(\omega t)\,\hat{x} + A\cos(\omega t)\,\hat{y}$$

This is a vector of magnitude A that rotates at a constant angular rate of ω in a counterclockwise sense. The vector traces out a circle. This is called circularly polarized.

At $z=0$



In general if we use complex exponential representations the polarization of the wave is described by $\tilde{\vec{A}}$. We then have certain special possibilities:

If

$$\tilde{\vec{A}} = \vec{A} e^{i\delta} = (A_x \hat{x} + A_y \hat{y}) e^{i\delta}$$

then the oscillation is along one line and the wave is linearly polarized. In this case δ is irrelevant. If $\delta_x \neq \delta_y$ and

$$\tilde{\vec{A}} = (A_x e^{i\delta_x} \hat{x} + A_y e^{i\delta_y} \hat{y})$$

then the wave is not linearly polarized. In general the vector will trace out an ellipse and the wave is elliptically polarized.

Classical waves in three dimensions

It is possible to have waves that propagate in two and three dimensions. The basic example involves a scalar quantity $f(\vec{r},t) = f(x,y,z,t)$ that satisfies:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

and a more compact version of this is

$$\boxed{\vec{\nabla}^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

This is the classical three dimensional wave equation. We will consider sinusoidal solutions first and then eventually generic solutions. Throughout this we use the notation

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

2 Three dimensional wave equation

Consider the three dimensional wave equation

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}.$$

a) Show that

$$f(\mathbf{r},t) = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$$

where $A > 0$ is a solution given that $\mathbf{k}$ and ω satisfy a particular condition. Describe the condition.

b) Suppose that $\mathbf{k} = k\hat{\mathbf{x}}$. Determine surfaces along which $f(\mathbf{r},t)$ attains a maximum. Describe the shape of these surfaces, the direction in which they propagate and the speed with which the propagate.

c) Suppose that $\mathbf{k} = k(\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$. Determine surfaces along which $f(\mathbf{r},t)$ attains a maximum. Describe the shape of these surfaces, the direction in which they propagate and the speed with which the propagate.

Answer: a) Need to check

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Then

$$\frac{\partial f}{\partial x} = -A\sin(\vec{k}\cdot\vec{r} - \omega t)\,\frac{\partial}{\partial x}(\vec{k}\cdot\vec{r} - \omega t)$$

$$= -A\sin(\vec{k}\cdot\vec{r} - \omega t)\,\frac{\partial}{\partial x}(k_x x + k_y y + k_z z - \omega t)$$

$$= -A k_x \sin(\vec{k}\cdot\vec{r} - \omega t)$$

Then:

$$\frac{\partial^2 f}{\partial x^2} = -A k_x \cos(\vec{k}\cdot\vec{r} - \omega t)\,\frac{\partial}{\partial x}(k_x x + k_y y + k_z z - \omega t)$$

$$= -A k_x^2 \cos(\vec{k}\cdot\vec{r} - \omega t)$$

$$= -k_z^2 f$$

Similarly we get

$$\frac{\partial^2 f}{\partial y^2} = -k_y^2 f \qquad \frac{\partial^2 f}{\partial z^2} = -k_z^2 f$$

$$\frac{\partial^2 f}{\partial t^2} = -\omega^2 f$$

Assembling these gives

$$-k_x^2 f - k_y^2 f - k_z^2 f = \frac{1}{v^2}(-\omega^2) f$$

$$\Rightarrow \quad k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{v^2} \qquad \Rightarrow \quad \vec{k}\cdot\vec{k} = \omega^2/v^2$$

$$\Rightarrow \quad \omega^2 = v^2\, \vec{k}\cdot\vec{k}$$

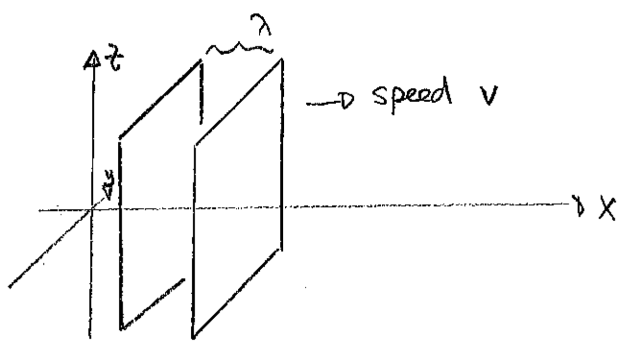
$$\omega = v\sqrt{\vec{k}\cdot\vec{k}}$$

b) $f(\vec{r},t) = A\cos(kx - \omega t)$

max when $kx - \omega t = 0, 2\pi, 4\pi$ etc....

$$\Rightarrow \quad x = \frac{\omega}{k}t + \frac{2\pi}{k}n \qquad \Rightarrow \quad x = vt + \frac{2\pi}{k}n$$

These are surfaces along which $x = \text{constant}$ at any instant, i.e. planes parallel to the yz plane. They are separated by $\Delta x = \frac{2\pi}{k}$ $= \lambda$



Wave travels along $+x$ with speed v

c) Here we need $\vec{k}\cdot\vec{r} - \omega t = 2n\pi$

Then $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\vec{k}\cdot\vec{r} = \frac{k}{\sqrt{2}}(x+y)$$

gives

$$\frac{k}{\sqrt{2}}(x+y) = \omega t + 2n\pi$$

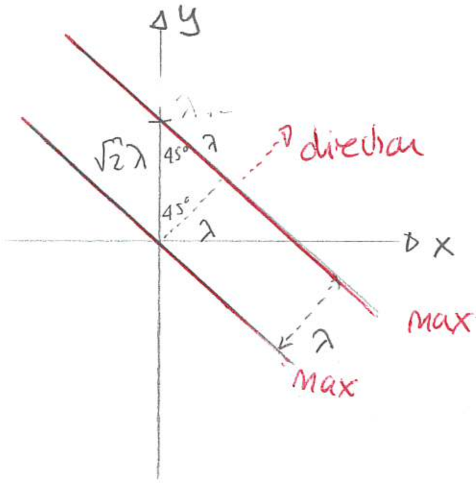
$$\Rightarrow \quad x + y = \sqrt{2}\,\frac{\omega}{k}\,t + \frac{2n\pi}{k}\sqrt{2}$$

$$\Rightarrow \quad y = -x + \sqrt{2}\,\frac{\omega}{k}\,t + \frac{2n\pi}{k}\sqrt{2}$$

$$\Rightarrow \quad y = -x + \sqrt{2}\,v\,t + n\lambda\sqrt{2}$$

These are planes parallel to z and perpendicular to the xy plane

At $t=0$, $\Rightarrow$, As t increases the slope of these curves stays the same. The intercept advances with speed $\sqrt{2}v$. This means the wavefronts advance with speed v in the direction $\frac{1}{\sqrt{2}}(\hat{x}+\hat{y})$



The exercise illustrates two examples of plane waves, solutions where the surfaces of constant f are planes.

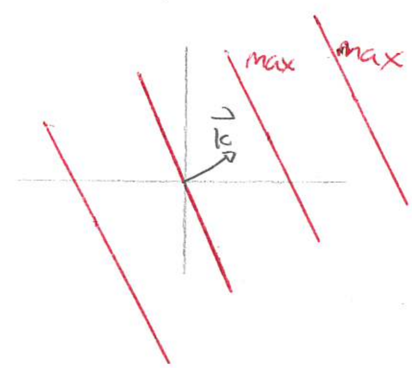
The general plane sinusoidal wave solution to the three dimensional wave equation is

$$f(\vec{r},t) = A\cos(\vec{k}\cdot\vec{r} - \omega t + \delta)$$

where A, ω, δ are constants (usually positive) and $\vec{k}$ is a vector. This has properties

1) $\omega = kv = \sqrt{\vec{k}\cdot\vec{k}}\, v$

2) the surfaces along which f is constant are planes perpendicular to $\vec{k}$



3) the distance between successive maximal planes is $\lambda = 2\pi/k$

4) the disturbance propagates along $\vec{k}$ with speed $v = \omega/k$.

The complex representation of these is

$$\boxed{\tilde{f}(\vec{r},t) = \tilde{A}e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad \text{where} \quad \tilde{A} = Ae^{i\delta}}$$

The generic plane wave solution to the three dimensional wave equation is

$$f(\vec{r},t) = g(\vec{k}\cdot\vec{r} - \omega t) + h(\vec{k}\cdot\vec{r} + \omega t)$$

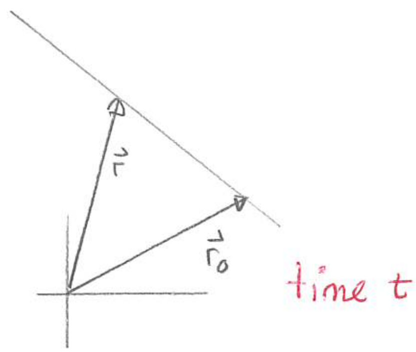
where g, h are arbitrary functions of a single variable

We can prove this as follows. Consider $g(\vec{k}\cdot\vec{r}-\omega t)$. This returns a single constant value whenever

$$\vec{k}\cdot\vec{r}-\omega t = \alpha = \text{constant}.$$

$$\Rightarrow \vec{k}\cdot\vec{r} = \omega t+\alpha$$

This is the equation of a plane perpendicular to $\vec{k}$. The reason is that. Let $\vec{r}_0$ be any vector to this plane. Then let $\vec{r}$ be a vector to an arbitrary point. So



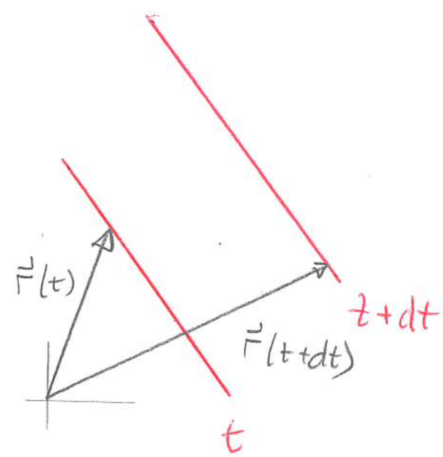
$$(\vec{r}-\vec{r}_0)\cdot\vec{k}=0$$

$$\Rightarrow \vec{r}\cdot\vec{k} = \underbrace{\vec{r}_0\cdot\vec{k}}_{\text{constant}}$$

$$\Rightarrow \quad xk_x+yk_y+zk_z = \underbrace{\vec{r}_0\cdot\vec{k}}_{\text{constant}}$$

co-ords of any pt on plane.

Now consider how the plane propagates from time $t \to t+dt$. Let $\vec{r}(t)$ be a vector to a point on the plane at time t. Let $\vec{r}(t+dt)$ be a vector to a point on the plane at time $t+dt$. So



$$\vec{r}(t+dt)\cdot\vec{k} = \omega(t+dt)+\alpha$$

$$\vec{r}(t)\cdot\vec{k} = \omega t+\alpha$$

$$\Rightarrow \left[\vec{r}(t+dt)-\vec{r}(t)\right]\cdot\vec{k} = \omega\, dt$$

Let $\Delta\vec{r} = \vec{r}(t+dt)-\vec{r}(t)$. $\Rightarrow (\Delta\vec{r})\cdot\vec{k} = \omega\, dt$

The distance between the planes is the smallest $\Delta \vec{r}$ that satisfies this. Then let θ be the angle between $\Delta \vec{r}$ and $\vec{k}$

$$\Delta r\, k \cos\theta = \omega\, dt \implies \Delta r = \frac{\omega\, dt}{k \cos\theta}$$

Δr is minimized when $\cos\theta$ is maximized. This occurs when $\cos\theta = 1$

Thus $\theta = 0$. So the wave travels parallel to $\vec{k}$. The speed is

$$\lim \frac{\Delta r}{dt} = \omega/k = v$$

The wave travels with velocity

$$\vec{v} = v\hat{k} = \omega/k\, \hat{k}$$

where $\hat{k}$ is a unit vector along $\vec{k}$.