

Tues. Exam 1

Covers Ch 4, 6, 7.3, 8.1

HW 1-8

2012, 2016 Exam 1.

Bring 1/2-Single letter sheet single side

Given: Front / back covers of text.Electric Stress Tensor

We have been able to prove that:

The net force on the material within a region R can be computed from the electric fields at the surface of the region using:



$$\vec{F}_{\text{mech}} = \left(\oint_S \vec{T}_x \cdot d\vec{a} \right) \hat{x} + \left(\oint_S \vec{T}_y \cdot d\vec{a} \right) \hat{y} + \left(\oint_S \vec{T}_z \cdot d\vec{a} \right) \hat{z}$$

where the electric stress vectors are

$$\vec{T}_x = \epsilon_0 (E_x E_x - \frac{1}{2} \vec{E} \cdot \vec{E}) \hat{x} + \epsilon_0 E_y E_x \hat{y} + \epsilon_0 E_z E_x \hat{z}$$

$$\vec{T}_y = \epsilon_0 E_x E_y \hat{x} + \epsilon_0 (E_y E_y - \frac{1}{2} \vec{E} \cdot \vec{E}) \hat{y} + \epsilon_0 E_z E_y \hat{z}$$

$$\vec{T}_z = \epsilon_0 E_x E_z \hat{x} + \epsilon_0 E_y E_z \hat{y} + \epsilon_0 (E_z E_z - \frac{1}{2} \vec{E} \cdot \vec{E}) \hat{z}$$

and $\vec{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$

We can arrange these in a matrix or tensor: (electric stress tensor)

$$\underline{\underline{T}} = \epsilon_0 \begin{pmatrix} (E_x E_x - \frac{1}{2} \vec{E} \cdot \vec{E}) & E_x E_y & E_x E_z \\ E_y E_x & (E_y E_y - \frac{1}{2} \vec{E} \cdot \vec{E}) & E_y E_z \\ E_z E_x & E_z E_y & (E_z E_z - \frac{1}{2} \vec{E} \cdot \vec{E}) \end{pmatrix}$$

Note for example that if $d\vec{a} = da_x \hat{x} \leadsto \begin{pmatrix} da_x \\ 0 \\ 0 \end{pmatrix}$

then

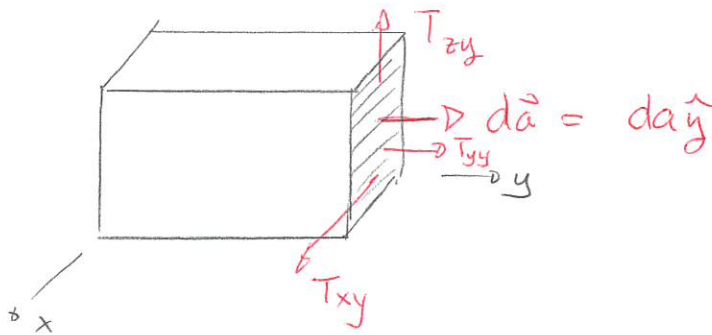
$$\underline{\underline{T}} \cdot d\vec{a} = \epsilon_0 \begin{pmatrix} (E_x E_x - \frac{1}{2} \vec{E} \cdot \vec{E}) da_x \\ (E_y E_x) da_x \\ (E_z E_x) da_x \end{pmatrix}$$

$$\text{and } \oint_S \underline{\underline{T}} \cdot d\vec{a} = \begin{pmatrix} \epsilon_0 \oint \vec{T}_x \cdot d\vec{a} \\ \epsilon_0 \oint \vec{T}_y \cdot d\vec{a} \\ \epsilon_0 \oint \vec{T}_z \cdot d\vec{a} \end{pmatrix}$$

In general the Maxwell electric stress tensor will be symmetric and has the form

$$\underline{\underline{T}} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

We can consider a region with rectangular faces.



On the illustrated face

$$\underline{\underline{T}}_x \cdot d\vec{a} = T_{xy} \quad \leadsto \text{force component along } x \quad (\text{shear})$$

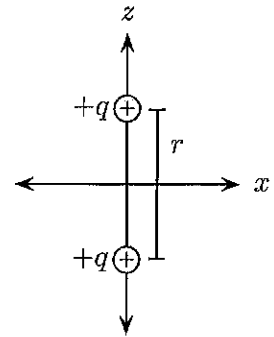
$$\underline{\underline{T}}_y \cdot d\vec{a} = T_{yy} \quad \leadsto \text{ " " " " } y$$

$$\underline{\underline{T}}_z \cdot d\vec{a} = T_{zy} \quad \leadsto \text{ " " " " } z$$

Thus some of these are shear forces, others are stresses.

1 Electric Stress Tensor

Consider two positive point charges along the z axis, Each has charge $+q$ and one is located at $z = +r/2$ while the other is at $z = -r/2$. The aim of this exercise is to use the electric stress tensor over a hemisphere with infinite radius and whose base lies in the xy plane.

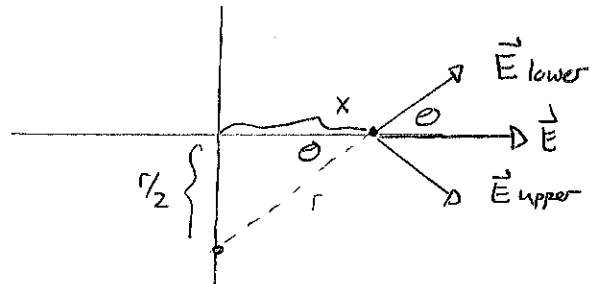


- Determine the electric field produced by both charges along the xy plane.
- Determine the electric stress tensor along the xy plane.
- Use the electric stress tensor to determine the force exerted on all the matter in the $z > 0$ region.

Answer:

- a) Consider a location along the x -axis. Then from the lower charge

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{(x^2 + r^2/4)}$$



gives the magnitude. The horizontal component is $E \cos \theta$ and

$$\cos \theta = \frac{x}{\sqrt{x^2 + r^2/4}}$$

Thus the horizontal component from one charge is:

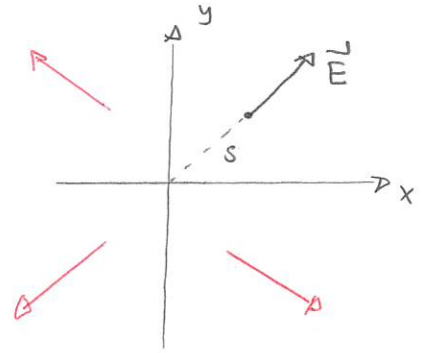
$$E = \frac{qx}{4\pi\epsilon_0 (x^2 + r^2/4)^{3/2}}$$

and at this location both charges contribute

$$\vec{E} = \frac{2qx}{4\pi\epsilon_0 (x^2 + r^2/4)^{3/2}} \hat{x}$$

The situation is symmetrical about the z -axis. Thus at distance s from the origin:

$$\vec{E} = \frac{2qs}{4\pi\epsilon_0 (s^2 + r^2/4)^{3/2}} \hat{s}$$



We then get $\hat{s} = \cos\phi \hat{x} + \sin\phi \hat{y}$. So

$$\vec{E} = \frac{1}{2\pi\epsilon_0} \left(\frac{qs}{(s^2 + r^2/4)^{3/2}} \right) [\cos\phi \hat{x} + \sin\phi \hat{y}]$$

b) Clearly $E_z = 0$. Also

$$\vec{E} \cdot \vec{E} = \left(\frac{1}{2\pi\epsilon_0} \right)^2 \frac{(qs)^2}{(s^2 + r^2/4)^3}$$

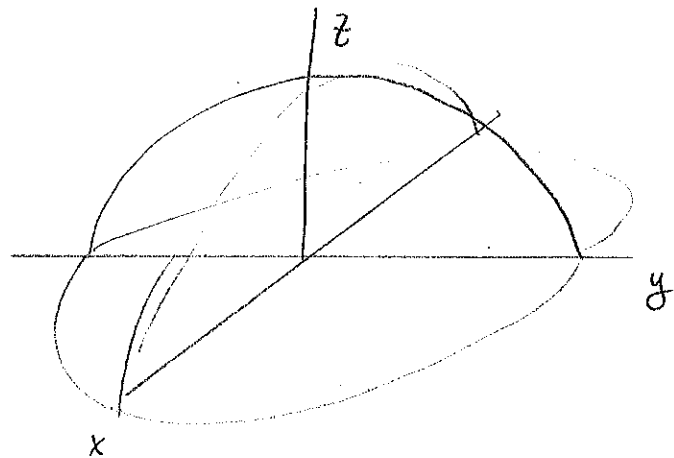
Thus:

$$T = \frac{\epsilon_0}{(2\pi\epsilon_0)^2} \frac{q^2 s^2}{(s^2 + r^2/4)^3} \begin{pmatrix} \cos^2\phi - \frac{1}{2} & \cos\phi \sin\phi & 0 \\ \cos\phi \sin\phi & \sin^2\phi - \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

c) We use a hemisphere

Then we need

$$\vec{F} = \left(\oint \vec{T}_x \cdot d\vec{a} \right) \hat{x} \\ + \left(\oint \vec{T}_y \cdot d\vec{a} \right) \hat{y} \\ + \left(\oint \vec{T}_z \cdot d\vec{a} \right) \hat{z}$$



over the whole surface. As the curved part $\rightarrow \infty$ this gives zero.
So we need to integrate over the base.

$$0 < s \leq \infty$$

$$0 \leq \phi \leq 2\pi$$

$$d\vec{a} = -s ds d\phi \hat{z} \quad \text{outward!}$$

We use:

$$\vec{T}_x = \frac{\epsilon_0}{(2\pi\epsilon_0)^2} \frac{q^2 s^2}{(s^2 + r^2/4)^3} \left[(\cos^2\phi - 1/2) \hat{x} + \cos\phi \sin\phi \hat{y} \right]$$

$$\vec{T}_y = \frac{\epsilon_0}{(2\pi\epsilon_0)^2} \frac{q^2 s^2}{(s^2 + r^2/4)^3} \left[\cos\phi \sin\phi \hat{y} + (\sin^2\phi - 1/2) \hat{x} \right]$$

$$\vec{T}_z = \frac{\epsilon_0}{(2\pi\epsilon_0)^2} \frac{q^2 s^2}{(s^2 + r^2/4)^3} \left(-\frac{1}{2} \right) \hat{z}$$

Clearly only $\vec{T}_z \cdot d\vec{a} \neq 0$. Thus

$$\vec{F} = + \frac{\epsilon_0 q^2}{2(2\pi\epsilon_0)^2} \int_0^\infty \frac{s^2}{(s^2 + r^2/4)^3} s ds \int_0^{2\pi} d\phi \hat{z} \\ = + \frac{q^2}{4\pi\epsilon_0} \int_0^\infty \frac{s^3}{(s^2 + r^2/4)^3} ds \hat{z}$$

The relevant "u substitution" is

$$s = r/2 \tan \theta \quad \Rightarrow \quad \frac{ds}{d\theta} = \frac{r}{2} \frac{1}{\cos^2 \theta}$$

$$\Rightarrow \vec{F} = + \frac{q^2}{4\pi\epsilon_0} \left(\frac{r}{2}\right)^4 \int_0^{\pi/2} \frac{\tan^3 \theta}{\left(\frac{r}{2}\right)^6 (\tan^2 \theta + 1)^3 \cos^2 \theta} d\theta \quad \hat{z}$$

$$= - \frac{q^2}{4\pi\epsilon_0} \left(\frac{2}{r}\right)^2 \int_0^{\pi/2} \frac{\sin^3 \theta / \cos^2 \theta}{\cos^2 \theta \left(\frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta}\right)^3} d\theta \quad \hat{z}$$

$$= \frac{+q^2}{4\pi\epsilon_0 r^2} 4 \int_0^{\pi/2} \cos \theta \sin^3 \theta d\theta \quad \hat{z}$$

$$= + \frac{q^2}{4\pi\epsilon_0 r^2} 4 \left(\frac{1}{4}\right) \sin^4 \theta \Big|_0^{\pi/2} \hat{z}$$

$$\vec{F} = \frac{q^2}{4\pi\epsilon_0 r^2} \hat{z}$$

This is exactly what Coulomb's Law predicts.

Momentum conservation

In the most general non-static situation we found that the net force on all the material in a region is

$$\begin{aligned}\vec{F}_{\text{mech}} = & -\epsilon_0 \frac{d}{dt} \int \vec{E} \times \vec{B} d\tau \\ & + \int \left\{ \epsilon_0 (\vec{\nabla} \cdot \vec{E}) \vec{E} - \epsilon_0 \vec{E} \times (\vec{\nabla} \times \vec{E}) \right. \\ & \left. + \frac{1}{\mu_0} (\vec{\nabla} \cdot \vec{B}) \vec{B} - \frac{1}{\mu_0} \vec{B} \times (\vec{\nabla} \times \vec{B}) \right\} d\tau\end{aligned}$$

We now need to incorporate the terms such as $\vec{E} \times (\vec{\nabla} \times \vec{E})$. We know that

$$\vec{\nabla} (\vec{E} \cdot \vec{E}) = \vec{E} \times (\vec{\nabla} \times \vec{E}) + \vec{E} \times (\vec{\nabla} \times \vec{E}) + (\vec{E} \cdot \vec{\nabla}) \vec{E} + (\vec{E} \cdot \vec{\nabla}) \vec{E}$$

Thus

$$\vec{E} \times (\vec{\nabla} \times \vec{E}) = \frac{1}{2} \vec{\nabla} (\vec{E} \cdot \vec{E}) - (\vec{E} \cdot \vec{\nabla}) \vec{E}$$

Thus:

$$\begin{aligned}& \epsilon_0 (\vec{\nabla} \cdot \vec{E}) \vec{E} - \epsilon_0 \vec{E} \times (\vec{\nabla} \times \vec{E}) \\ &= \epsilon_0 (\vec{\nabla} \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \vec{\nabla}) \vec{E} + \frac{1}{2} \vec{\nabla} (\vec{E} \cdot \vec{E})\end{aligned}$$

Thus

$$\begin{aligned}\vec{F}_{\text{mech}} = & -\epsilon_0 \frac{d}{dt} \int (\vec{E} \times \vec{B}) d\tau + \epsilon_0 \int \left[(\vec{\nabla} \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \vec{\nabla}) \vec{E} - \frac{1}{2} \vec{\nabla} (\vec{E} \cdot \vec{E}) \right] d\tau \\ & + \frac{1}{\mu_0} \int \left[(\vec{\nabla} \cdot \vec{B}) \vec{B} + (\vec{B} \cdot \vec{\nabla}) \vec{B} - \frac{1}{2} \vec{\nabla} (\vec{B} \cdot \vec{B}) \right] d\tau\end{aligned}$$

To simplify this we consider components. For example the x component of the electric field term:

$$\begin{aligned} & (\vec{\nabla} \cdot \vec{E}) E_x + (\vec{E} \cdot \vec{\nabla}) E_x - \frac{1}{2} \frac{\partial}{\partial x} (\vec{E} \cdot \vec{E}) \\ &= \frac{\partial}{\partial x} (E_x^2 - \frac{1}{2} \vec{E} \cdot \vec{E}) + \frac{\partial}{\partial y} (E_x E_y) + \frac{\partial}{\partial z} (E_x E_z) \end{aligned}$$

Proof: $(\vec{\nabla} \cdot \vec{E}) E_x = \frac{\partial E_x}{\partial x} E_x + \frac{\partial E_y}{\partial y} E_x + \frac{\partial E_z}{\partial z} E_x$

$$(\vec{E} \cdot \vec{\nabla}) E_x = E_x \frac{\partial E_x}{\partial x} + E_y \frac{\partial E_x}{\partial y} + E_z \frac{\partial E_x}{\partial z}$$

Adding these and subtracting $\frac{1}{2} \frac{\partial}{\partial x} (\vec{E} \cdot \vec{E})$ gives

$$2 E_x \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} E_x + \frac{\partial E_z}{\partial z} E_x$$

$$+ E_y \frac{\partial E_x}{\partial y} + E_z \frac{\partial E_x}{\partial z} - \frac{1}{2} \frac{\partial E_x^2}{\partial x} - \frac{1}{2} \frac{\partial E_y^2}{\partial x} - \frac{1}{2} \frac{\partial E_z^2}{\partial x}$$

$$= \frac{\partial}{\partial x} (E_x^2 - \frac{1}{2} \vec{E} \cdot \vec{E}) + \frac{\partial}{\partial y} (E_y E_x) + \frac{\partial}{\partial z} (E_z E_x) \quad \square$$

Similarly

$$(\vec{\nabla} \cdot \vec{B}) B_x + (\vec{B} \cdot \vec{\nabla}) B_x - \frac{1}{2} \frac{\partial}{\partial x} (\vec{B} \cdot \vec{B})$$

$$= \frac{\partial}{\partial x} (B_x^2 - \frac{1}{2} \vec{B} \cdot \vec{B}) + \frac{\partial}{\partial y} (B_y B_x) + \frac{\partial}{\partial z} (B_z B_x)$$

Thus we define the Maxwell stress vectors:

$$\begin{aligned}\vec{T}_x &:= \epsilon_0 \left(E_x^2 - \frac{1}{2} \vec{E} \cdot \vec{E} \right) \hat{x} + \epsilon_0 E_y E_x \hat{y} + \epsilon_0 E_z E_x \hat{z} \\ &\quad + \frac{1}{\mu_0} \left(B_x^2 - \frac{1}{2} \vec{B} \cdot \vec{B} \right) \hat{x} + \frac{1}{\mu_0} B_y B_x \hat{y} + \frac{1}{\mu_0} B_z B_x \hat{z}\end{aligned}$$

$$\begin{aligned}\vec{T}_y &= \epsilon_0 E_x E_y \hat{x} + \epsilon_0 \left(E_y^2 - \frac{1}{2} \vec{E} \cdot \vec{E} \right) \hat{y} + \epsilon_0 E_z E_y \hat{z} \\ &\quad + \frac{1}{\mu_0} B_x B_y \hat{x} + \frac{1}{\mu_0} \left(B_y^2 - \frac{1}{2} \vec{B} \cdot \vec{B} \right) \hat{y} + \frac{1}{\mu_0} B_z B_y \hat{z}\end{aligned}$$

$$\vec{T}_z = \dots$$

Then this gives:

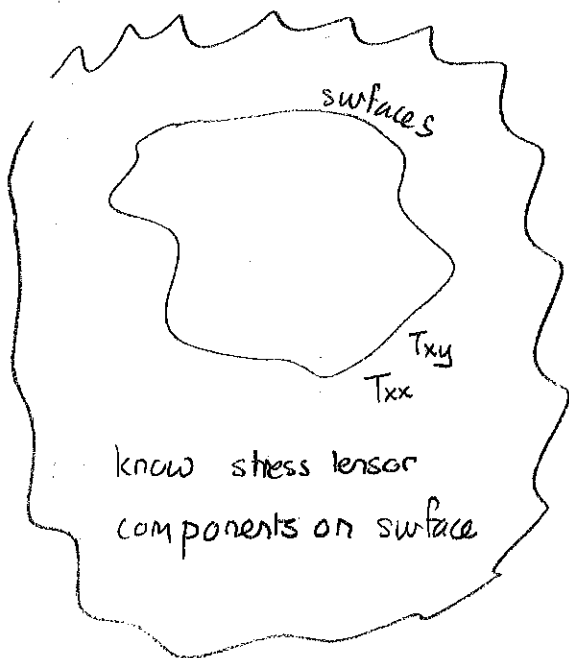
$$\begin{aligned}\vec{F}_{\text{mech}} &= -\epsilon_0 \frac{d}{dt} \int (\vec{E} \times \vec{B}) d\tau \\ &\quad + \int_R (\vec{\nabla} \cdot \vec{T}_x) d\tau \hat{x} + \int_R (\vec{\nabla} \cdot \vec{T}_y) d\tau \hat{y} + \int_R (\vec{\nabla} \cdot \vec{T}_z) d\tau \hat{z}\end{aligned}$$

$$\begin{aligned}\Rightarrow \vec{F}_{\text{mech}} &= -\epsilon_0 \frac{d}{dt} \int_R (\vec{E} \times \vec{B}) d\tau \\ &\quad + \left(\oint_S \vec{T}_x \cdot d\vec{a} \right) \hat{x} + \left(\oint_S \vec{T}_y \cdot d\vec{a} \right) \hat{y} + \left(\oint_S \vec{T}_z \cdot d\vec{a} \right) \hat{z}\end{aligned}$$

Again we can create a matrix that represents the Maxwell stress tensor:

$$\underline{\underline{T}} = \begin{pmatrix} \epsilon_0(E_x^2 - \frac{1}{2}\vec{E}\cdot\vec{E}) + \frac{1}{\mu_0}(B_x^2 - \frac{1}{2}\vec{B}\cdot\vec{B}) & \epsilon_0 E_x E_y + \frac{1}{\mu_0} B_x B_y & \epsilon_0 E_x E_z + \frac{1}{\mu_0} B_x B_z \\ \epsilon_0 E_y E_x + \frac{1}{\mu_0} B_y B_x & \epsilon_0(E_y^2 - \frac{1}{2}\vec{E}\cdot\vec{E}) + \frac{1}{\mu_0}(B_y^2 - \frac{1}{2}\vec{B}\cdot\vec{B}) & \epsilon_0 E_y E_z + \frac{1}{\mu_0} B_y B_z \\ \epsilon_0 E_z E_x + \frac{1}{\mu_0} B_z B_x & \epsilon_0 E_z E_y + \frac{1}{\mu_0} B_z B_y & \epsilon_0(E_z^2 - \frac{1}{2}\vec{E}\cdot\vec{E}) + \frac{1}{\mu_0}(B_z^2 - \frac{1}{2}\vec{B}\cdot\vec{B}) \end{pmatrix}$$

We get the conceptual scheme



Net force on matter inside the surface is

$$\begin{aligned} \vec{F}_{\text{mech}} = & -\epsilon_0 \frac{d}{dt} \int (\vec{E} \times \vec{B}) dz \\ & + \left(\oint_S \vec{T}_x \cdot d\vec{a} \right) \hat{x} \\ & + \dots \end{aligned}$$

Finally we have that $\vec{F}_{\text{mech}} = \frac{d\vec{P}_{\text{mech}}}{dt}$. Thus, if we define the electromagnetic momentum as:

$$\vec{P}_{\text{elec}} = \epsilon_0 \int_{\text{region}} \vec{E} \times \vec{B} \, d\tau = \epsilon_0 \mu_0 \int_{\text{region}} \vec{S} \, d\tau$$

We get a rule for the conservation of momentum:

$$\frac{d}{dt} (\vec{P}_{\text{mech}} + \vec{P}_{\text{elec}}) = \left(\oint_S \vec{T}_x \cdot d\vec{a} \right) \hat{x} + \left(\oint_S \vec{T}_y \cdot d\vec{a} \right) \hat{y} + \left(\oint_S \vec{T}_z \cdot d\vec{a} \right) \hat{z}$$

This gives conservation over all space in the case where $\vec{E}, \vec{B} \rightarrow 0$ as $r \rightarrow \infty$ because the surface integrals approach zero. But the conservation law now necessarily includes an electromagnetic component. The electromagnetic momentum density is then:

$$\vec{g} = \epsilon_0 (\vec{E} \times \vec{B}) = \mu_0 \epsilon_0 \vec{S}$$

Here $\vec{S}$ is the Poynting vector.