Electromagnetic Theory II: Homework 22

Due: 10 May 2021

1 Proper time

Various particles that move in one dimension are observed from an inertial frame. The positions of these are recorded with respect to time as measured in this frame.

a) The position of particle A is

$$x = \frac{4}{5}ct$$

Determine an expression for the proper time of this particle in terms of t. Does proper time appear to run at a constant rate as observed in this frame?

b) The position of particle B is

$$x = \frac{mc^2}{F}\left[\sqrt{1+\left(\frac{Ft}{mc}\right)^2} - 1\right].$$

Determine an expression for the proper time of this particle in terms of t. Does proper time appear to run at a constant rate as observed in this frame? *This is the trajectory of a particle with mass m that moves with constant relativistic force, F.*

2 Transformation of ordinary velocities via proper velocities

The ordinary classical velocity of a particle as observed from the S frame is $\mathbf{u}$. The same particle is observed from the S' frame, which moves with velocity $\mathbf{v} = v\hat{\mathbf{x}}$ as observed from the unprimed frame. The aim of this exercise is to use the proper velocity in both frames to determine the ordinary velocity as observed from the S'.

a) Determine an expression for the proper velocity in the unprimed frame in terms of the ordinary velocity in that frame. Use this to determine an expression for the proper velocity in the primed frame in terms of the ordinary velocity in the unprimed frame.

b) Use the previous result to relate ordinary velocity as observed from the S' frame to that as observed in the S frame.

3 Time, proper time and position in relativity

A particle moves with (ordinary) constant velocity $\mathbf{u} = \frac{3}{5}c\hat{\mathbf{x}}$ as observed from the frame S and at $t = 0$ it is at $x = 0$. The frame S' moves along the x axis with velocity $\frac{4}{5}c$ relative to the S frame; the origins of these frames coincide at $t = 0$. The aim of this exercise is to determine the location particle at various times and proper times in the two frames and to compare these locations to see if they are the same. First consider ordinary times. *In this exercise you should write all velocities as multiples of c and all distances should be kept in units of light seconds ($c\,$s).*

a) Show that the ordinary velocity of the particle in the S' frame is $\mathbf{u}' = -\frac{5}{13}\, c\hat{\mathbf{x}}$.

b) Consider the moment at $t = 0$ in the S frame. Determine the location of the particle in the S' frame. What is the value of t' at this instant? Do $t = 0$ and $t' = 0$ refer to the same spacetime point on the particle's trajectory?

c) Determine the particle's location as observed at $t = 2\,\mathrm{s}$ in the S frame. Determine its location in the S' frame. Does this agree with the location as observed in the S' frame at $t' = 2\,\mathrm{s}$? Do $t = 2\,\mathrm{s}$ and $t' = 2\,\mathrm{s}$ refer to the same spacetime point on the particle's trajectory?

Now consider proper times.

d) Determine a relationship between the proper time, τ, and the time, t, as measured in the S frame. Use this to determine the particle's location and time when $\tau = 2\,\mathrm{s}$ as observed from the S frame.

e) Determine a relationship between the proper time, τ, and the time, t, as measured in the S' frame. Use this to determine the particle's location and time when $\tau = 2\,\mathrm{s}$ as observed from the S' frame.

We now aim to compare the particle locations at the same proper time in each frame. The coordinates for the locations will generally be different in the two frames. However, if the observers compare their results and find that they match according to the Lorentz transforms, i.e. $x'^1 = \gamma(x^1 - \beta x^0)$, then they agree that the locations are identical.

f) Use the Lorentz transform to compare the location of the particle as observed in the S frame at proper time $\tau = 2\,\mathrm{s}$ to that as observed in the S' frame at proper time $\tau = 2\,\mathrm{s}$. Are the locations identical?

4 Two point charges

Two point charges are at rest in the S frame. The source, with charge q is at the origin and the test, with charge Q is at $\mathbf{r} = y\hat{\mathbf{y}}$.

a) In the S frame, determine the fields produced by the source and the force that these exert on the test.

Let S' be a frame moving with velocity $\mathbf{v}$ left along the $-x$ axis.

b) In the S' frame, determine the electric and magnetic fields produced by the point source charge and the force $\mathbf{F}'$ that this exerts on the test charge. Assume that the source is a point charge with magnitude q to do this.

c) Use the relativistic force transformations to determine $\mathbf{F}'$ from $\mathbf{F}$. Does this agree with the result as obtained via the fields?

5 Relativistic force, acceleration and fields

The relativistic force and ordinary acceleration are related via:

$$\mathbf{F} = \frac{m}{\sqrt{1 - u^2/c^2}} \left[\mathbf{a} + \frac{1}{1 - u^2/c^2} \frac{(\mathbf{u} \cdot \mathbf{a})}{c^2} \mathbf{u} \right]$$

where $\mathbf{u}$ is the ordinary velocity and $u^2 = \mathbf{u} \cdot \mathbf{u}$.

a) Show that

$$\mathbf{u} \cdot \mathbf{a} = \frac{(1 - u^2/c^2)^{3/2}}{m} \mathbf{u} \cdot \mathbf{F}$$

and use the result to show that

$$\mathbf{a} = \frac{\sqrt{1 - u^2/c^2}}{m} \left[\mathbf{F} - \frac{(\mathbf{u} \cdot \mathbf{F})}{c^2} \mathbf{u} \right]$$

b) Assuming that the Lorentz force law gives the relativistic force, show that

$$\mathbf{a} = \frac{q}{m} \sqrt{1 - u^2/c^2} \left[\mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{(\mathbf{u} \cdot \mathbf{E})}{c^2} \mathbf{u} \right]$$

where $\mathbf{E}$ and $\mathbf{B}$ are the fields produced by sources.

c) Does the modified acceleration still result in circular orbital motion for a particle in an electric field?

d) Suppose that the sources only produce a uniform constant magnetic field and at one instant the particle's ordinary velocity is perpendicular to the magnetic field. Describe the subsequent motion of the charged particle.