Phys 312
Fall 2016

Electromagnetic Theory II: Final Exam

14 December 2016

Name: SOLUTION **Total:** /75

Instructions

- There are 7 questions on 13 pages.
- Show your reasoning and calculations and always explain your answers.

For maximum credit you need to complete five out of the seven questions. If you attempt all seven questions you will be given the five highest scores that you attain on individual questions.

Physical constants and useful formulae

Permittivity of free space	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$
Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$
Charge of a proton	$e = +1.60 \times 10^{-19}\,\mathrm{C}$
Speed of light	$c = 3.0 \times 10^{8}\,\mathrm{m/s}$

Question 1

A particular charge distribution produces potentials (in spherical coordinates)

$$V = 0$$
$$\mathbf{A} = \frac{A_0}{r}\sin(\omega t)\hat{\phi}$$

where A_0 is a constant.

a) Determine the electric and magnetic fields that this produces.

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$= -\vec{\nabla} 0 - \frac{\partial}{\partial t} \frac{A_0}{r} \sin(\omega t)\, \hat{\phi}$$

$$\vec{E} = -\frac{A_0 \omega}{r} \cos(\omega t)\, \hat{\phi}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$ (only $A_\phi = \frac{A_0}{r} \sin(\omega t)$ is non-zero)

$$= \frac{1}{r\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta A_\phi\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(r A_\phi\right)\hat{\theta}$$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{A_0 \sin\theta}{r}\sin(\omega t)\right]\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left[\cancel{A_0 \sin(\omega t)}\right]\hat{\theta}$$

$$= \frac{A_0}{r^2 \sin\theta} \cos\theta \sin(\omega t)\, \hat{r}$$

$$\vec{B} = \frac{A_0 \cos\theta}{r^2 \sin\theta} \sin(\omega t)\, \hat{r}$$

Question 1 continued ...

b) Determine the charge distributions that produce these fields.

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \text{and} \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

For the electric field only $E_\phi \neq 0$. Thus

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(-\frac{A_0\omega}{r}\cos(\omega t)\right) = 0$$

$$\Rightarrow \boxed{\rho = 0}$$

Then $$\mu_0\vec{J} = \vec{\nabla}\times\vec{B} - \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \vec{J} = \frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{B}\right) - \epsilon_0\frac{\partial\vec{E}}{\partial t}$$

For $\vec{\nabla}\times\vec{B}$ only $B_r \neq 0$. Thus

$$\vec{\nabla}\times\vec{B} = \cancel{\frac{1}{r}\frac{1}{\sin\theta}\frac{\partial B_r}{\partial\phi}}\,\hat{\theta} - \frac{1}{r}\frac{\partial B_r}{\partial\theta}\hat{\phi}$$

$$= -\frac{1}{r}\left(\frac{A_0}{r^2}\right)\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{\sin\theta}\right)\sin(\omega t)\,\hat{\phi} = +\frac{A_0}{r^3}\frac{\sin\omega t}{\sin^2\theta}\,\hat{\phi}$$

$$\frac{\partial\vec{E}}{\partial t} = +\frac{A_0\omega^2}{r}\sin(\omega t)\,\hat{\phi}$$

$$\Rightarrow \vec{J} = -\frac{A_0\sin(\omega t)}{r}\left[\epsilon_0\omega^2 - \frac{1}{r^2\mu_0\sin^2\theta}\right]\hat{\phi}$$

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$$= -\frac{A_0\sin(\omega t)}{\mu_0 r}\left[\frac{\omega^2}{c^2} - \frac{1}{r^2\sin^2\theta}\right]\hat{\phi}$$

Question 2

Consider the potentials, given in cylindrical coordinates as

$$V = -\alpha \ln s$$

$$\mathbf{A} = \frac{\alpha t}{s}\,\hat{\mathrm{s}} + \beta \sin(\omega t)\hat{\phi}$$

where α and β are constants.

a) Show that this potential is in the Coulomb gauge.

Coulomb gauge requires $\vec{\nabla}\cdot\vec{A} = 0$

Here $A_s = \frac{\alpha t}{s}$ $A_\phi = \beta \sin(\omega t)$ and

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\cancel{\frac{\partial A_\phi}{\partial \phi}} + \cancel{\frac{\partial A_z}{\partial z}}$$

$$= \frac{1}{s}\frac{\partial}{\partial s}[\alpha t]$$

$$= 0$$

This is the Coulomb gauge.

b) Let

$$\lambda = -\alpha t \ln s$$

generate a gauge transformation. Determine the potentials in this new gauge. Is this in either the Lorentz or the Coulomb gauge?

With a gauge transformation $V \to V' = V - \frac{\partial \lambda}{\partial t}$

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\lambda$$

Question 2 continued ...

Thus

$$V' = -\alpha \ln s - \frac{\partial}{\partial t}\left(-\alpha t \ln s\right)$$

$$= -\alpha \ln s + \alpha \ln s$$

$$\boxed{V' = 0}$$

Then

$$\vec{A}' = \frac{\alpha t}{s}\hat{s} + \beta \sin(\omega t)\hat{\phi} + \vec{\nabla}\left(-\alpha t \ln s\right)$$

$$= \frac{\alpha t}{s}\hat{s} + \beta \sin(\omega t)\hat{\phi} - \alpha t \underbrace{\vec{\nabla}[\ln s]}$$

$$= \frac{\partial}{\partial s}(\ln s)\hat{s}$$

$$= \frac{1}{s}\hat{s}$$

$$= \cancel{\frac{\alpha t}{s}\hat{s}} + \beta \sin(\omega t)\hat{\phi} - \cancel{\frac{\alpha t}{s}\hat{s}}$$

$$\vec{A}' = \beta \sin(\omega t)\hat{\phi}$$

Now $\vec{\nabla}\cdot\vec{A} = 0$ by the previous calculation

and $\frac{\partial V}{\partial t} = 0$

Thus $\vec{\nabla}\cdot\vec{A} + \frac{1}{\mu_0\epsilon_0}\frac{\partial V}{\partial t} = 0$

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Lorentz gauge ✓

Coulomb gauge ✓

Question 3

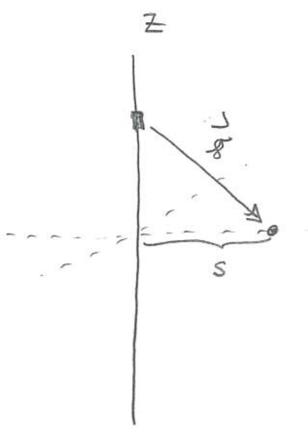
An infinitely long wire lies along the z axis and carries current

$$I = \begin{cases} 0 & \text{if } t < 0 \\ I_0 e^{t/\tau} & \text{if } t > 0 \end{cases}$$

where $\tau > 0$ is a constant with units of time. The charge density along the wire is always zero.

a) Suppose that the potentials (and eventually fields) are to be determined at a point in the xy plane a distance s from the wire. Indicate which portions of the current in the wire contribute to the vector potential at this location at time $t > 0$ and use these to *set up* an integral for the field *(you do not need to evaluate the integral)*.

↳ in terms of s, t, etc, ...



$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}', t_r)}{\mathscr{r}}\, d\tau'$$

where $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

$$t_r = t - \mathscr{r}/c.$$

Here $\vec{r} = s\hat{s}$, $\vec{r}' = z'\hat{z}$ $\Rightarrow$ $\mathscr{r} = \sqrt{s^2 + z'^2}$

Then for a one dimensional current along $\hat{z}$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{I(z', t_r)}{\mathscr{r}}\, dz'\, \hat{z}$$

Now we need $t_r > 0$ for a contribution. Thus

$$t - \mathscr{r}/c > 0 \Rightarrow tc > \mathscr{r}$$

$$\Rightarrow (tc)^2 > s^2 + z'^2$$

$$\Rightarrow z'^2 < (tc)^2 - s^2$$

only contributions from

$$-\sqrt{(tc)^2 - s^2} < z' < \sqrt{(tc)^2 - s^2}$$

b) Explain *how* you could determine the electric field produced by the current.

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a) Cont....

$$\vec{A}(\vec{r}',t) = \frac{\mu_0}{4\pi} \int \frac{I(z', t_r)}{\mathscr{r}} dz' \hat{z}$$

$$= \frac{\mu_0}{4\pi} \int_{-\sqrt{(ct)^2-s^2}}^{+\sqrt{\quad}} \frac{I_0 e^{t_r/\tau}}{\mathscr{r}} dz' \hat{z}$$

$$= \frac{\mu_0 I_0}{4\pi} e^{t/\tau} \int_{-\sqrt{c^2t^2-s^2}}^{\sqrt{c^2t^2-s^2}} \frac{e^{-\sqrt{s^2+z'^2}/\tau c}}{\sqrt{s^2+z'^2}} dz' \hat{z}$$

b) $\vec{E} = -\cancel{\vec{\nabla} V} - \frac{\partial \vec{A}}{\partial t}$

$= -\frac{\partial \vec{A}}{\partial t}$

Differentiate $\vec{A}$ w.r.t. t.

Question 4

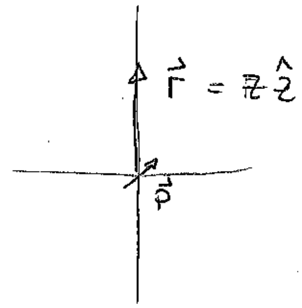
An electric dipole in the xy plane oscillates in such a way that the dipole moment is

$$\mathbf{p} = p_0 \left[\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}}\right]$$

where p_0 is the magnitude of the dipole moment and ω is a constant frequency.

a) Determine expressions for the radiation electric and magnetic fields at any point along the z axis.

$$\vec{E} = \frac{\mu_0}{4\pi r}\left[\hat{r} \times (\hat{r} \times \ddot{\vec{p}})\right]$$



$$\vec{B} = -\frac{\mu_0}{4\pi r c}\left[\hat{r} \times \ddot{\vec{p}}\right]$$

With $\vec{r} = z\hat{z}$ we get

$$\vec{E} = \frac{\mu_0}{4\pi z}\left[\hat{z} \times (\hat{z} \times \ddot{\vec{p}})\right] = \frac{\mu_0}{4\pi z}\left[\hat{z}\,(\hat{z}\cdot\ddot{\vec{p}}) - \ddot{\vec{p}}\,\underbrace{\cancel{(\hat{z}\cdot\hat{z})}}_{1}\right]$$

$$\vec{B} = \frac{-\mu_0}{4\pi z c}\left[\hat{z} \times \ddot{\vec{p}}\right]$$

Now $\quad \ddot{\vec{p}} = -\omega^2 p_0 \left[\cos(\omega t)\,\hat{x} + \sin(\omega t)\,\hat{y}\right]$

Then $\quad \hat{z}\cdot\ddot{\vec{p}} = 0$ gives:

$$\vec{E} = -\frac{\mu_0}{4\pi z}\ddot{\vec{p}}$$

$$\Rightarrow \vec{E} = \frac{\mu_0 p_0 \omega^2}{4\pi z}\left[\cos(\omega t)\,\hat{x} + \sin(\omega t)\,\hat{y}\right]$$

Question 4 continued ...

$$\vec{B} = \frac{-\mu_0}{4\pi z c}\left(-\omega^2 p_0\right)\Big[\cos(\omega t)\ \underbrace{\hat{z}\times\hat{x}}_{\hat{y}} + \sin(\omega t)\ \underbrace{\hat{z}\times\hat{y}}_{-\hat{x}}\Big]$$

$$\vec{B} = \frac{\mu_0\omega^2 p_0}{4\pi z c}\left[-\sin(\omega t)\,\hat{x} + \cos(\omega t)\,\hat{y}\right]$$

b) Determine an expression for the total power radiated by the dipole (in all directions).

$$P_{rad} = \frac{\mu_0}{6\pi c}\left[\ddot{p}(t_0)\right]^2$$

at $t_0 = t - r/c$

$$\text{Now}\ \left[\ddot{p}\right]^2 = \omega^4 p_0^2\left[\sin^2(\omega t) + \cos^2(\omega t)\right]$$

$$= \omega^4 p_0^2$$

$$\Rightarrow \quad P_{rad} = \frac{\mu_0 p_0^2 \omega^4}{6\pi c.}$$

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Question 5

Consider a classical electron, with mass m, that orbits around a fixed proton in a circular orbit with radius r.

a) Let U be the total energy of the particle. Show that the rate at which the energy changes is

$$\frac{dU}{dt} = -\frac{\mu_0 e^6}{96 c \pi^3 \epsilon_0^2 m^2} \frac{1}{r^4}$$

The power radiated is

$$P = \frac{\mu_0 q^2}{6\pi c} a^2$$

where a is acceleration. Now $P = -\frac{dU}{dt}$

$$\Rightarrow \frac{dU}{dt} = -\frac{\mu_0 q^2}{6\pi c} a^2$$

$\downarrow \vec{F}_{elec}$

$\ominus$ +

Now $\vec{F}_{net} = m\vec{a}$

$$\Rightarrow F_{elec} = ma \Rightarrow \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2} = a\,m$$

$$\Rightarrow a = \frac{1}{4\pi\epsilon_0}\frac{q^2}{m r^2}$$

$$\Rightarrow \frac{dU}{dt} = \frac{-\mu_0 q^4}{96\pi^2 c\, \epsilon_0^2 m^2}\frac{1}{r^4}$$

b) Explain *how* you could use this to determine the lifetime of a classical atom.

Express $U = K + U_{elec}$ in terms of r

$$= \frac{1}{2} m v^2 - \frac{1}{4\pi\epsilon_0}\frac{q^2}{r}$$

$a = \frac{v^2}{r} \Rightarrow v^2 = r\left(\frac{q^2}{4\pi\epsilon_0 m r^2}\right.$

$$= -\frac{1}{8\pi\epsilon_0}\frac{q}{r}$$

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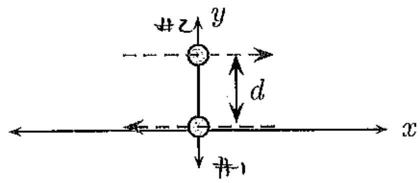
Then $\frac{dU}{dt}$ contains $\frac{dr}{dt}$ gives a diff eqn.

$\frac{dt}{dr} = \ldots$

solve for $t \to$ evaluate at $(r = r_0)$ $(r = 0)$.

Question 6

Two identical charged particles, each with charge $+q$, move with the same constant speeds, v in opposite directions along lines parallel to the x axis. At one instant the charges are situated as illustrated and the questions below refer to this instant.



a) Determine the fields that the charge at the origin produces at the location of the other charge.

Charge moves with velocity $\vec{v} = -v\hat{x}$

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{(1-v^2/c^2)}{(1-\frac{v^2}{c^2}\sin^2\theta)^{3/2}} \frac{\hat{R}}{R^2} \qquad \vec{R} = d\hat{y}$$

where $\vec{R}$ is indicated vector. θ is the angle between $\vec{R}$ and $\vec{v}$. So $\theta = \pi/2$. Thus,

$$\vec{E} = \frac{q}{4\pi\epsilon_0 d^2 \sqrt{1-v^2/c^2}} \hat{y}$$

Now $\vec{B} = \frac{1}{c^2}\vec{v}\times\vec{E}$

$$= \frac{1}{c^2} \frac{q}{4\pi\epsilon_0 d^2} \frac{v}{\sqrt{1-v^2/c^2}} \underbrace{(-\hat{x}\times\hat{y})}_{-\hat{z}}$$

$$\vec{B} = -\frac{v}{c^2} \frac{q}{4\pi\epsilon_0 d^2} \frac{1}{\sqrt{1-v^2/c^2}} \hat{z}$$

Question 6 continued ...

b) Determine the force exerted by the charge at the origin on the other charge.

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$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$$

where $\vec{u} = v\hat{x}$ is the velocity of the upper particle.

So

$$\vec{F} = \frac{q^2}{4\pi\epsilon_0 d^2} \frac{1}{\sqrt{1-v^2/c^2}} \left\{ \hat{y} - \frac{v}{c^2} (v\hat{x}) \times \hat{z} \right\}$$

$$\vec{F} = \frac{q^2}{4\pi\epsilon_0 d^2} \frac{1+v^2/c^2}{\sqrt{1-v^2/c^2}} \hat{y}$$

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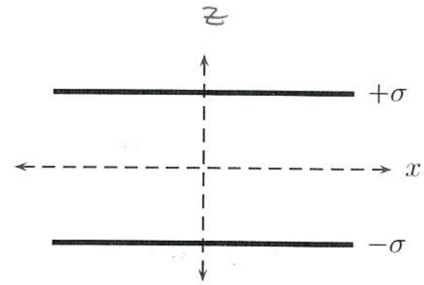
Question 7



The electric field produced by a uniformly charged infinite sheet of *stationary charge* is

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\,\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a normal vector perpendicular to and away from the sheet and σ is the charge per unit area. Consider two parallel sheets, where the upper sheet has charge density σ and the lower sheet has charge density $-\sigma$. Both sheets *move* with constant speed v along the $+x$ axis. Determine the electric and magnetic fields at all locations.

a) Determine fields in rest frame.

$\downarrow E_{upper} = -\frac{\sigma}{2\epsilon_0}\hat{z}$ $\quad \downarrow \vec{E}_{lower} = -\frac{\sigma}{2\epsilon_0}\hat{z}$ $\quad \Rightarrow \quad \vec{E} = -\frac{2\sigma}{2\epsilon_0}\hat{z}$

$\downarrow \vec{E}_{upper}$ $\quad \uparrow \vec{E}_{lower}$ cancel. $\qquad \vec{E} = -\frac{\sigma}{\epsilon_0}\hat{z}$

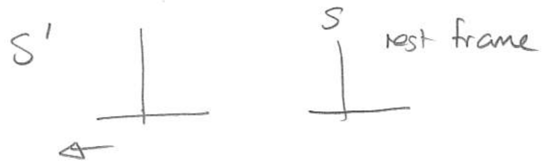
Thus

Between plates: $\vec{E} = -\frac{\sigma}{\epsilon_0}\hat{z}$ $\quad \vec{B} = 0$ (no currents and static charges)

Outside ,, $\vec{E} = 0$ $\quad \vec{B} = 0$

Question 7 continued . . .

b) ~~Use~~ Determine fields in frame where plates move right with ~~velocity $\vec{v}$~~ speed v



$$\vec{v} = -v\hat{x}$$

$$E_x' = \cancel{E_x} \qquad B_x' = \cancel{B_x}$$

$$E_y' = \gamma(\cancel{E_y} + v\cancel{B_z}) \qquad B_y' = \gamma\left(\cancel{B_y} - \frac{v}{c^2}E_z\right)$$

$$E_z' = \gamma(E_z - v\cancel{B_y}) \qquad B_z' = \gamma\left(\cancel{B_z} + \frac{v}{c^2}\cancel{E_y}\right)$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

Thus $$\vec{E}' = \frac{1}{\sqrt{1-v^2/c^2}}\left(-\frac{\sigma}{\epsilon_0}\right)\hat{z}$$

$$\vec{B}' = \frac{1}{\sqrt{1-v^2/c^2}}\left(\frac{v}{c^2}\right)\frac{\sigma}{\epsilon_0}\,\hat{y}$$

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