Phys 231 2021

Lecture 39

Weds: Read 8.4

Thurs: HW by 5pm → D2L

Multiple particles in quantum physics

There are many situations in which there are multiple particles present. We need to extend quantum theory to these situations. Examples include:

1) electrons in atoms

Li (3 electrons)

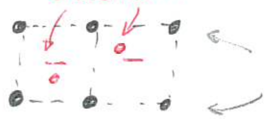
- need to have some wavefunction that describes the three electrons

2) multiple trapped ions

- need wavefunction that describes all three ions.

3) electrons in a conductor

electrons



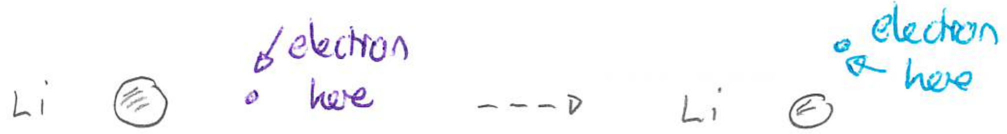
lattice of nuclei

- need wavefunction to describe all electrons.

There will be two fundamentally distinct situations in cases where multiple particles are present:

1) particles are distinguishable - for example the row of trapped ions where each ion is restricted to a region and the ions can be identified by their location

2) particles are indistinguishable - for example electrons in an atom



observe at one moment

observe at another moment

→ Is this the same electron??

The resulting wavefunctions will turn out to have fundamentally different properties with implications in various physical situations such as the structure of atoms.

Two particles in an infinite well

As an example consider two particles in a one-dimensional infinite well. Suppose that the particles are distinguishable. For example:

Particle 1 $\equiv$ neutron $\rightarrow$ mass m_1

Particle 2 $\equiv$ proton $\rightarrow$ mass m_2

We could measure the position of each particle and record the results using two co-ordinates:

x_1 $\rightarrow$ outcome of measurement for particle 1 $\equiv$ co-ordinate for particle 1

x_2 $\rightarrow$ " " " " " 2 $\equiv$ co-ordinate for particle 2.

Then we expect to describe the <u>state of both particles</u> by a wavefunction

$$\Psi(x_1, x_2, t)$$

This will satisfy the Schrödinger equation:

$$\boxed{-\frac{\hbar^2}{2m_1}\frac{\partial^2 \Psi}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2 \Psi}{\partial x_2^2} + U(x_1, x_2)\,\Psi(x_1, x_2, t) = i\hbar \frac{\partial \Psi(x_1, x_2, t)}{\partial t}}$$

The comparable TISE describes a wavefunction $\psi(x_1, x_2)$:

$$\boxed{-\frac{\hbar^2}{2m_1}\frac{\partial^2 \psi}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2 \psi}{\partial x_2^2} + U(x_1, x_2)\,\psi(x_1, x_2) = E\,\psi(x_1, x_2)}$$

where E is the system energy.

For an infinite well $U = 0$ and thus:

$$\psi(x_1, x_2) = 0 \qquad \text{for} \qquad \begin{array}{l} x_1, x_2 \leq 0 \\ x_1, x_2 \geq L \end{array}$$

and within the well:

$$-\frac{\hbar^2}{2m_1}\frac{\partial^2 \psi}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2 \psi}{\partial x_2^2} = E\,\psi(x_1, x_2)$$

We need to solve this equation. Again a separation of variables strategy would yield that energy eigenstates for the pair of particles are constructed from products of energy eigenstates for the individual particles.

Possible energy eigenstate

$$\psi(x_1, x_2) = \underbrace{\psi_n(x_1)}_{\text{particle 1}}\ \underbrace{\psi_{n'}(x_2)}_{\text{particle 2}}$$

$\psi_n(x_1)$ is an energy eigenstate for particle 1, i.e.

$$-\frac{\hbar^2}{2m_1}\frac{d^2\psi_n}{dx_1^2} = E_n \psi_n$$

where $E_n = \dfrac{\hbar^2 \pi^2}{2m_1 L^2} n^2$

$\psi_n(x_2)$ is an energy eigenstate for particle 2

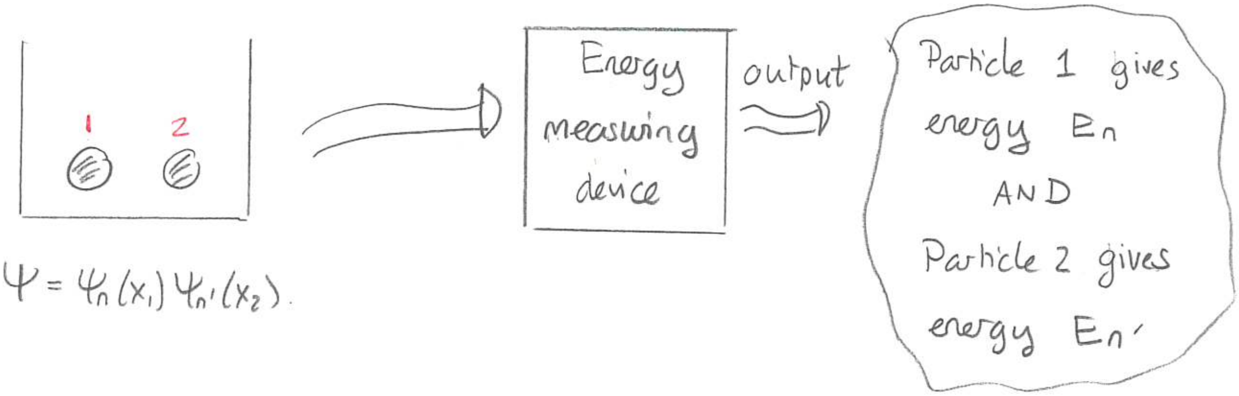
$$-\frac{\hbar^2}{2m_2}\frac{d^2\psi_{n'}}{dx_2^2} = E_{n'} \psi_{n'}$$

with energy $E_{n'} = \dfrac{\hbar^2 \pi^2}{2m_2 L^2} n'^2$

Energy for system is

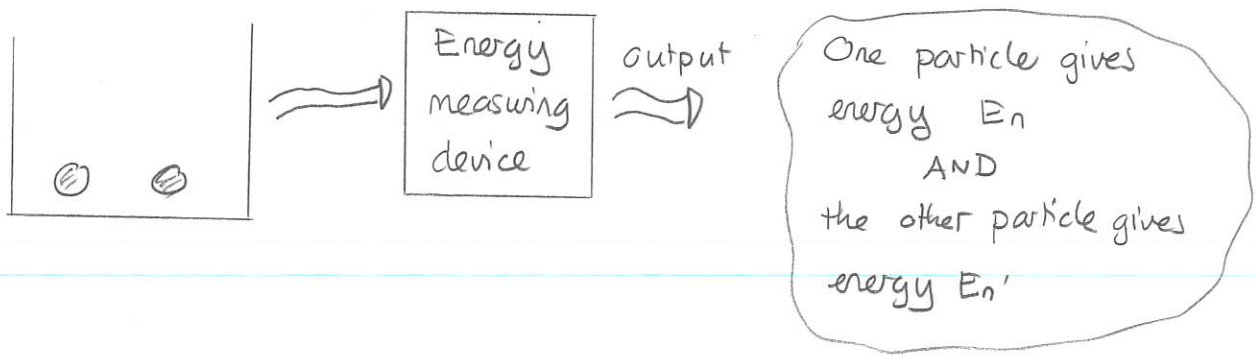
$$E = E_n + E_{n'}$$

In this case we can envisage energy measurements as:



$\Psi = \psi_n(x_1)\,\psi_{n'}(x_2).$

Now what if the particles are identical. Then we cannot tell which particle gives energy E_n and which gives energy $E_{n'}$. The energy measurement should give:



Clearly the state $\psi_n(x_1)\psi_{n'}(x_2)$ does not capture this situation as it ascribes energy E_n to particle 1 and $E_{n'}$ to particle 2 and they would then be identifiable.

Neither does the state $\psi_{n'}(x_1)\psi_n(x_2)$ capture the situation as it just reverses which particle is which.

We can try some combination of such states.

Superpositions of states

In general quantum theory allows for combinations of states.

If $\Psi_1(x,t)$ and $\Psi_2(x,t)$ each satisfy the time-dependent Schrödinger equation then any linear combination

$$\Psi(x,t) = \alpha \Psi_1(x,t) + \beta \Psi_2(x,t)$$

where α, β are complex constants such that $|\alpha|^2 + |\beta|^2 = 1$ also satisfies the Schrödinger equation

We can extend this to provide superpositions for multiple particle states. We would like to describe the situation:

An energy measurement yields E_n for one particle and $E_{n'}$ for the other particle and we cannot tell which particle yielded which outcome.

Given that state ψ_n is associated with energy E_n and $\psi_{n'}$ with energy $E_{n'}$ we can construct superpositions such as

$$\psi(x_1, x_2) = \alpha \psi_n(x_1)\psi_{n'}(x_2) + \beta \psi_{n'}(x_1)\psi_n(x_2)$$

where $|\alpha|^2 + |\beta|^2 = 1$. The only way that we will not be able to distinguish between the particles is if $|\alpha|^2 = |\beta|^2 = \frac{1}{2}$.

The two basic wavefunctions that reflect this choice are:

Symmetric:

$$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}} \psi_n(x_1)\psi_{n'}(x_2) + \frac{1}{\sqrt{2}} \psi_{n'}(x_1)\psi_n(x_2)$$

$\hookrightarrow \Psi(x_2, x_1) = \Psi(x_1, x_2)$

Antisymmetric

$$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}} \psi_n(x_1)\psi_{n'}(x_2) - \frac{1}{\sqrt{2}} \psi_{n'}(x_1)\psi_n(x_2)$$

$\hookrightarrow$ ~~$\Psi(x_2, x_1)$~~

$\Psi(x_2, x_1) = -\Psi(x_1, x_2)$

Bosons + Fermions

In general particles fall into one of two categories:

1) Bosons - wavefunction for identical particles is symmetric under interchange of particles:

$$\Psi(x_1, x_2) = \psi_n(x_1)\psi_n(x_2) \qquad \text{(same } n\text{)}$$

OR $\Psi(x, x_2) = \frac{1}{\sqrt{2}} \psi_n(x_1)\psi_{n'}(x_2) + \frac{1}{\sqrt{2}} \psi_{n'}(x_1)\psi_n(x_2)$ different n

- particles with whole integer spin $\leadsto$ photons
- particles can be in same state

2) Fermions - wavefunction for identical particles is antisymmetric under interchange

$$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}} \psi_n(x_1)\psi_{n'}(x_2) - \frac{1}{\sqrt{2}} \psi_{n'}(x_1)\psi_n(x_2)$$

- particles with half integer spin $\rightarrow$ protons, neutrons, electrons

- particles <u>cannot be</u> in same state