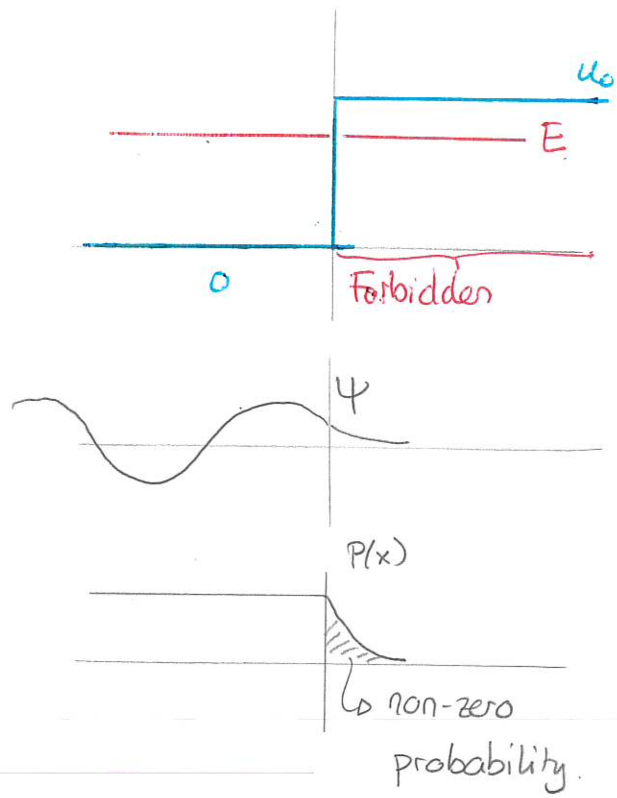
Mon: Read 7.1, 7.2

Tues: HW by 5pm

Potential Step

We saw that when the energy of an incident particle is less than the step there is a non-zero possibility for the particle to be located in the classically forbidden region. The extent to which the particle can penetrate the region depends on how close E is to U_0. We can again determine the reflection coefficient and we find that



Whenever $E < U_0$ the reflection coefficient is $R = 1$.

Demo PhET Quantum Tunnelling

– vary E or U_0 and observe penetration of wavefunction

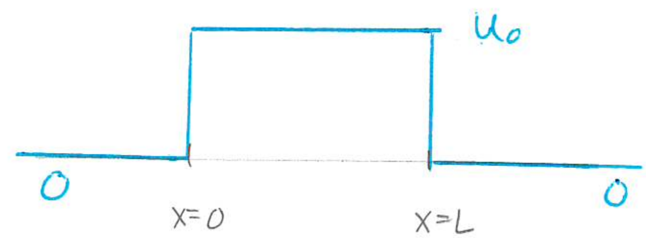
How can we observe such penetration into classically forbidden phenomena?

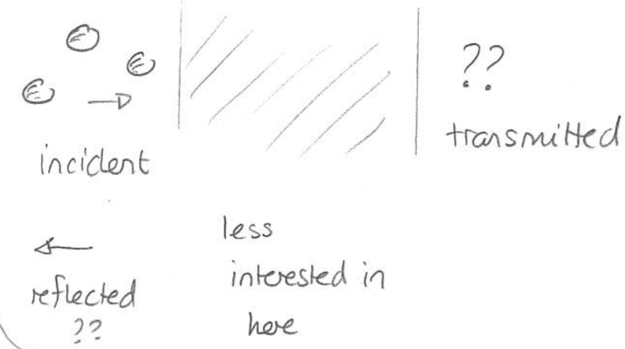
One way would be to construct a barrier of finite width.

Potential barrier

A simple version of a potential barrier consists of a rectangular step with finite width and height.

We again consider particles that are incident from the left. In classical physics the ultimate motion of the particles depends on the relationship of the energy to U_0.





Classical physics

$E > U_0$	$E < U_0$
particle definitely transmitted past barrier no chance of reflection	particle definitely reflected. No chance of transmission

The most interesting case in quantum theory is where $E < U_0$. We can assess this by:

1) solve the TISE in each region

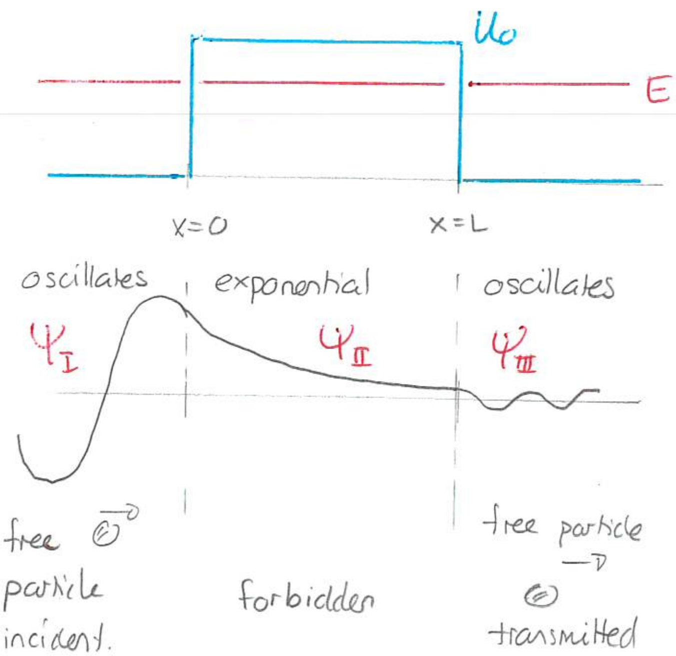
2) match at boundaries

$$\psi_I(0) = \psi_{II}(0)$$

$$\left.\frac{d\psi_I}{dx}\right|_0 = \left.\frac{d\psi_{II}}{dx}\right|_0$$

$$\psi_{II}(L) = \psi_{III}(L)$$

$$\left.\frac{d\psi_{II}}{dx}\right|_L = \left.\frac{d\psi_{III}}{dx}\right|_L$$



As before the solutions are:

<u>Region I (left of barrier)</u>

$$\Psi_I(x) = \underbrace{Ae^{ikx}}_{\substack{\text{incident} \\ \text{from left}}} + \underbrace{Be^{-ikx}}_{\text{reflected}} \qquad k = \sqrt{2mE/\hbar^2}$$

<u>Region II (inside barrier)</u>

$$\Psi_{II}(x) = Ce^{\alpha x} + De^{-\alpha x} \qquad \alpha = \sqrt{2m(U_0 - E)/\hbar^2}$$

<u>Region III (right of barrier)</u>

$$\Psi_{III}(x) = \underbrace{\cancel{Ee^{-ikx}}}_{\substack{\text{incident} \\ \text{from right} \\ \text{ignore}}} + \underbrace{Fe^{+ikx}}_{\text{transmitted}} \qquad k = \sqrt{2mE/\hbar^2}$$

We then need to apply matching conditions at $x=0$ and $x=L$

Quiz1 70%

~~[illegible]~~

We apply all four conditions and eventually we can obtain

$$F = (\text{expression involving } k, \alpha, L) \times A$$

$$B = (\text{expression } '' \quad k, \alpha, L) \times A$$

Then the reflection coefficient is:

$$R = \frac{k|B|^2}{k|A|^2} \Rightarrow R = \frac{|B|^2}{|A|^2}$$

and the transmission coefficient is

$$T = \frac{k|F|^2}{k|A|^2} \Rightarrow T = \frac{|F|^2}{|A|^2}$$

The analysis of the matching conditions gives:

$$\boxed{\begin{aligned} R &= \frac{(\alpha^2+k^2)^2 \sinh^2(\alpha L)}{(\alpha^2+k^2)^2 \sinh^2(\alpha L) + 4\alpha^2 k^2} \\ T &= \frac{4\alpha^2 k^2}{(\alpha^2+k^2)^2 \sinh^2(\alpha L) + 4\alpha^2 k^2} \end{aligned}}$$

The hyperbolic sin function is

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

Proof: At $x=0$ $\quad \psi_I(0) = \psi_{II}(0) \Rightarrow A+B = C+D$

$$\left.\frac{d\psi_1}{dx}\right|_0 = \left.\frac{d\psi_{II}}{dx}\right|_0 \Rightarrow ik(A-B) = \alpha(C-D)$$

At $x=L$ $\quad \psi_{II}(0) = \psi_{III}(0) \Rightarrow Ce^{\alpha L} + De^{-\alpha L} = Fe^{ikL}$

$$\left.\frac{d\psi_{II}}{dx}\right|_0 = \left.\frac{d\psi_{III}}{dx}\right|_0 \Rightarrow \alpha\left[Ce^{\alpha L} - De^{-\alpha L}\right] = ikFe^{ikL}$$

The first give:

$$\left.\begin{aligned} A+B &= C+D \\ \frac{ik}{\alpha}(A-B) &= C-D \end{aligned}\right\} \Rightarrow \quad \begin{aligned} C &= \frac{1}{2}\left[\left(1+\frac{ik}{\alpha}\right)A + \left(1-\frac{ik}{\alpha}\right)B\right] \\ D &= \frac{1}{2}\left[\left(1-\frac{ik}{\alpha}\right)A + \left(1+\frac{ik}{\alpha}\right)B\right] \end{aligned}$$

So

$$C = \frac{1}{2\alpha}\left[(\alpha+ik)A + (\alpha-ik)B\right]$$

$$D = \frac{1}{2\alpha}\left[(\alpha-ik)A + (\alpha+ik)B\right]$$

The last two give:

$$\left.\begin{aligned} C\ \alpha^{L} + De^{-\alpha L} &= Fe^{ikL} \\ C\ \alpha^{L} - De^{-\alpha L} &= \frac{ik}{\alpha}Fe^{ikL} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} 2Ce^{\alpha L} &= Fe^{ikL}\left(\frac{\alpha+ik}{\alpha}\right) \\ 2De^{-\alpha L} &= Fe^{ikL}\left(\frac{\alpha-ik}{\alpha}\right). \end{aligned}$$

$$\Rightarrow \quad C = Fe^{ikL}(\alpha+ik)\frac{1}{2\alpha}e^{-\alpha L}$$

$$D = Fe^{ikL}(\alpha-ik)\frac{1}{2\alpha}e^{\alpha L}$$

Thus:

$$Fe^{ikL}(\alpha+ik)\frac{1}{\cancel{2}\alpha}e^{-\alpha L} = \frac{1}{\cancel{2}\alpha}\left[(\alpha+ik)A + (\alpha-ik)B\right]$$

$$Fe^{+ikL}(\alpha-ik)\frac{1}{\cancel{2}\alpha}e^{\alpha L} = \frac{1}{\cancel{2}\alpha}\left[(\alpha-ik)A + (\alpha+ik)B\right]$$

$$\Rightarrow \quad (\alpha+ik)A + (\alpha-ik)B = Fe^{ikL}e^{-\alpha L}(\alpha+ik)$$

$$(\alpha-ik)A + (\alpha+ik)B = Fe^{+ikL}e^{\alpha L}(\alpha-ik)$$

Then we get:

$$(\alpha+ik)^2 A + (\alpha^2+k^2) B = (\alpha+ik)^2 e^{ikL} e^{-\alpha L} F$$
$$(\alpha-ik)^2 A + (\alpha^2+k^2) B = (\alpha-ik)^2 e^{ikL} e^{\alpha L} F$$

Subtracting gives:

$$\left[(\alpha+ik)^2 - (\alpha-ik)^2\right] A = e^{ikL} F \left\{ (\alpha^2-k^2)(e^{-\alpha L} - e^{\alpha L}) \quad 2\alpha ik\,(e^{-\alpha L} + e^{\alpha L}) \right\}$$

$$\Rightarrow \left[\alpha^2 + 2i\alpha k - k^2 - \alpha^2 + 2i\alpha k + k^2\right] A = e^{ikL} F \left\{ (\alpha^2-k^2)\, 2\sinh(\alpha L) + 2i\alpha k\, 2\cosh(\alpha L) \right\}.$$

$$\Rightarrow 4i\alpha k\, A = 2 e^{ikL} F \left\{ (\alpha^2-k^2)\sinh(\alpha L) + 2i\alpha k \cosh(\alpha L) \right\}.$$

$$\Rightarrow \frac{F}{A} = \frac{2i\alpha k\, e^{-ikL}}{(\alpha^2-k^2)\sinh\alpha L + 2i\alpha k\cosh(\alpha L)}$$

$$\Rightarrow \left|\frac{F}{A}\right|^2 = \frac{4\alpha^2k^2}{(\alpha^2-k^2)^2\sinh^2(\alpha L) + 4\alpha^2k^2(\sinh^2(\alpha L)+1)}$$

$\cosh^2\alpha L = 1 -$

$$= \frac{4\alpha^2k^2}{(\alpha^2+k^2)^2\sinh^2(\alpha L) + 4\alpha^2k^2}$$

Separately:
$$\frac{e^{\alpha L}}{(\alpha+ik)}\left[(\alpha+ik)A + (\alpha-ik)B\right] = \frac{e^{-\alpha L}}{(\alpha-ik)}\left[(\alpha-ik)A + (\alpha+ik)B\right].$$

$$\Rightarrow (e^{\alpha L} - e^{-\alpha L}) A = \left[\left(\frac{\alpha+ik}{\alpha-ik}\right)e^{-\alpha L} - \left(\frac{\alpha-ik}{\alpha+ik}\right)e^{\alpha L}\right] B$$

Now multiply by $(\alpha+ik)(\alpha-ik)$

$$\Rightarrow \quad \underbrace{(\alpha-ik)(\alpha+ik)}_{\alpha^2+k^2}\underbrace{(e^{\alpha L}-e^{-\alpha L})}_{2\sinh(\alpha L)}A = \left[(\alpha+ik)^2 e^{-\alpha L} - (\alpha-ik)^2 e^{\alpha L}\right]B$$

$$\Rightarrow \quad \cancel{2}(\alpha^2+k^2)\sinh(\alpha L)\; A = \left[(\alpha^2-k^2)\underbrace{(e^{-\alpha L}-e^{\alpha L})}_{-2\sinh\alpha L} + 2i\alpha k\underbrace{(e^{-\alpha L}+e^{+\alpha L})}_{2\cosh\alpha L}\right]B$$

$$\Rightarrow \quad B = \frac{(\alpha^2+k^2)\sinh(\alpha L)}{\left[-(\alpha^2-k^2)\sinh(\alpha L) + 2i\alpha k\cosh(\alpha L)\right]}A$$

$$\text{So} \quad |B|^2 = \frac{(\alpha^2+k^2)^2\sinh^2(\alpha L)}{\left[(\alpha^2-k^2)^2\sinh^2(\alpha L) + 4\alpha^2k^2\underbrace{\cosh^2(\alpha L)}_{\to\, 1+\sinh^2(\alpha L)}\right]^2}|A|^2$$

$$= \frac{(\alpha^2+k^2)^2\sinh^2(\alpha L)}{\left[(\alpha^2+k^2)^2\sinh^2(\alpha L) + 4\alpha^2k^2\right]}|A|^2$$

This gives the result for R

Demo: PhET Quantum Tunneling
- show tunneling

Now with
$$\alpha^2 = \frac{2m}{\hbar^2}(U_0 - E)$$
$$k^2 = \frac{2m}{\hbar^2}E$$

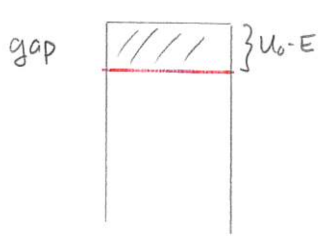
we get

$$\boxed{\begin{aligned} R &= \frac{\sinh^2(\alpha L)}{\sinh^2(\alpha L) + 4\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)} \\[2em] T &= \frac{4\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)}{\sinh^2(\alpha L) + 4\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)} \end{aligned}}$$

$$\alpha = \sqrt{2m(U_0-E)/\hbar^2}$$

This is still difficult to interpret since m, E, U_0 appear in various locations. The animation does indicate that this is extremely sensitive to barrier width. One possible method for interpreting this is the wide barrier approximation. Note that

$$\alpha^2 = \frac{2m}{\hbar^2}(U_0 - E) \Rightarrow U_0 - E = \frac{\hbar^2}{2m}\alpha^2 = \underbrace{\frac{\hbar^2}{2mL^2}}_{\text{basic energy infinite well.}}(\alpha L)^2$$



So $(\alpha L)^2$ measures the gap in terms of this basic energy.

We consider $\alpha L \gg 1$. Then

$$\sinh(\alpha L) = \frac{e^{\alpha L} - e^{-\alpha L}}{2} \approx \frac{e^{\alpha L}}{2}$$

This gives the <u>wide barrier approximation</u>

$$\boxed{T \approx 16 \frac{E}{U_0}\left(1 - \frac{E}{U_0}\right) e^{-2L\sqrt{2m(U_0 - E_0)/\hbar^2}} \qquad \alpha L \gg 0}$$

Quiz 2

Here $T = 16 \frac{1}{2}\frac{1}{2} e^{-2L\sqrt{\quad}}$

$$= 4e^{-2L\sqrt{\quad}}$$

Then $L' = 2L$ gives

$$T' = 4e^{-4L\sqrt{\quad}} = 4\left(e^{-2L\sqrt{\quad}}\right)^2 = 4\left(\frac{T}{4}\right)^2 = \frac{1}{4}T^2$$

Thus

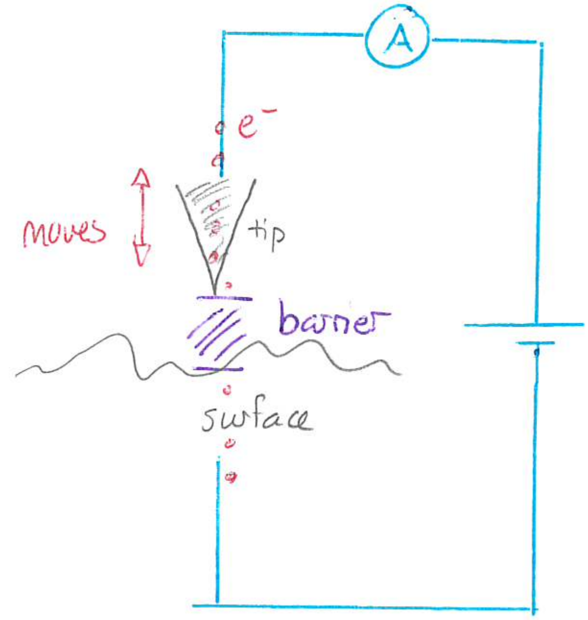
$$T' = \frac{1}{4}\left(\frac{1}{10}\right)^2 = \frac{1}{400}$$

This extreme sensitivity to barrier width is typical of tunneling in quantum theory.

Applications

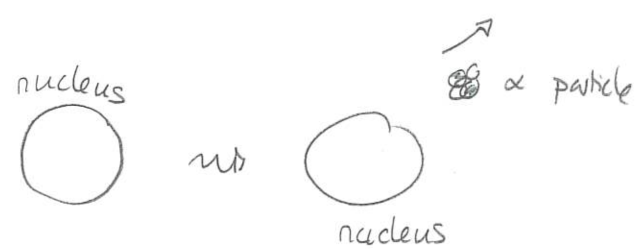
Show: nanoScience page

1) Scanning Tunneling Microscopy

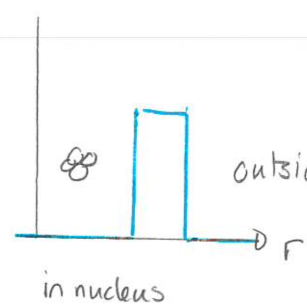


In scanning tunneling microscopy a tip hovers above a surface and a simple circuit is connected to the tip. The gap between the tip and surface constitutes a barrier and some electrons can tunnel. The rate at which they do so and the associated current are very sensitive to the gap. The tip is moved up and down by fractions of nanometers so that the current stays constant. In this way the tip traces the surface profile and maps the surface.

2) Radioactive decay



In radioactive decay a nucleus partly disintegrates. One way to model this is that the particle must tunnel through a barrier. The barrier widths vary appreciably for various nuclei giving a wide range of tunneling, and hence, decay rates. We get



$^{222}_{86}Rn$ half life = 3.8 days

$^{214}_{82}Po$ 3.1 min

$^{226}_{88}Ra$ 1600 yr