Phys 231 2021

Lecture 29

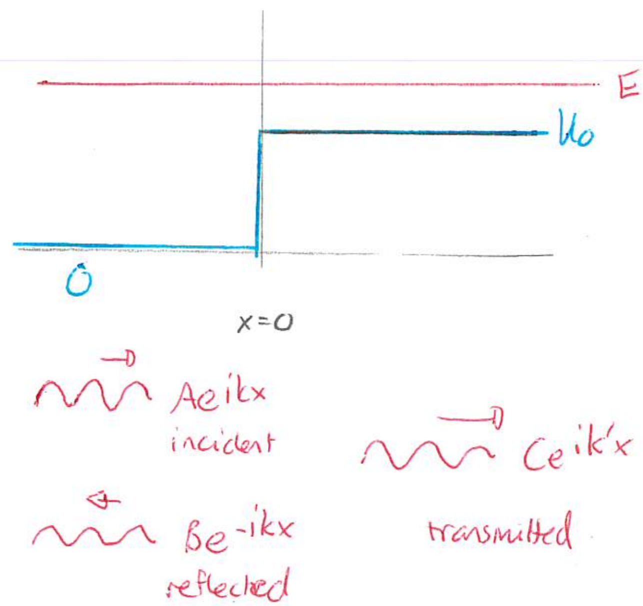
Weds: HW by 5pm

Fri: Read 6.2

Tues: Next HW

Step potential: energy larger than step

We consider a stationary state for the step potential in the case where $E > U_0$. We consider particles incident from the left only. We then find that the solutions to the TISE are:



1) for $x < 0$

$$\Psi(x) = Ae^{ikx} + Be^{-ikx}$$

$$k = \sqrt{2mE/\hbar^2}$$

2) for $x > 0$

$$\Psi(x) = Ce^{ik'x}$$

$$k' = \sqrt{2m(E-U_0)/\hbar^2}$$

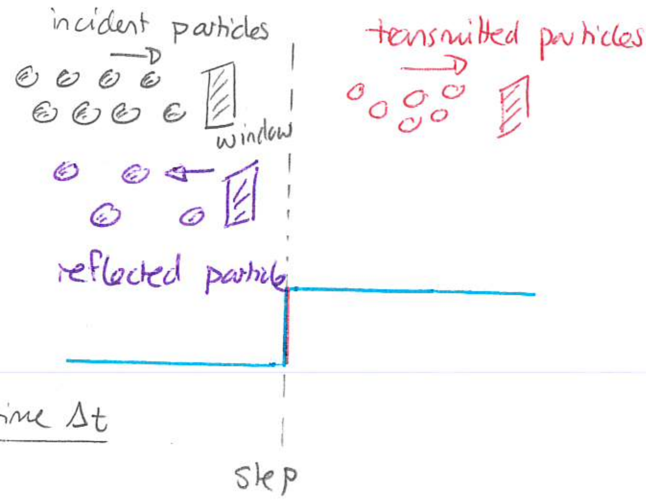
Matching at the boundary gives:

$$A + B = C$$
$$A - B = \frac{k'}{k} C$$

$$\Rightarrow \quad B = \frac{k-k'}{k+k'} A \qquad C = \frac{2k}{k+k'} A$$

The quantity B must describe the extent to which particles are reflected back. The quantity C must describe the extent to which particles are transmitted beyond the step. The correct use and interpretation of these arise by considering the rates at which incident particles are reflected and transmitted.

We imagine observing these through windows. The rate at which particles pass depends on the size of the window and we define



$$\text{rate} = \frac{\text{number of particles passing in time } \Delta t}{\Delta t}$$

In such rate calculations we expect:

1) rate is proportional to $|\psi|^2$ = amplitude of relevant wave

2) rate is proportional to speed $\Rightarrow$ rate proportional to p

$\Rightarrow$ rate " to $k = p/\hbar$

Thus we get:

rate at which particles are incident $=$ constant $|A|^2 k$ ← wavenumber left of step

rate at which particles are reflected $=$ constant $|B|^2 k$

rate at which particles are transmitted $=$ constant $|C|^2 k'$ ← wavenumber right of step

We can then define the reflection coefficient as

$$\boxed{R = \frac{\text{rate at which particles are reflected}}{\text{rate at which particles are incident}}}$$

The transmission coefficient is

$$\boxed{T = \frac{\text{rate at which particles are transmitted}}{\text{rate at which particles are incident}}}$$

1 Reflection and transmission a ta step

A particle with energy E is incident on a step potential, for which $U_0 < E$.

a) Determine expressions for the reflection and transmission coefficients in terms of $k = \sqrt{2mE/\hbar^2}$ and $k' = \sqrt{2m(E-U_0)/\hbar^2}$.

b) What do you expect that the sum of the two coefficients should equal? Verify that this is true using your expressions from the previous part.

c) Suppose that the reflection coefficient is $1/\cancel{2}4$. Determine the ratio E/U_0.

d) Given that E and U_0 are fixed what effect will increasing the mass of the particles have on the transmission coefficient?

a)

$$R = \frac{\cancel{\text{const}}\ |B|^2 \cancel{k}}{\cancel{\text{const}}\ |A|^2 \cancel{k}} = \frac{|B|^2}{|A|^2}$$

But

$$B = \left(\frac{k-k'}{k+k'}\right) A \Rightarrow \frac{B}{A} = \left(\frac{k-k'}{k+k'}\right)$$

$$\Rightarrow R = \left(\frac{k-k'}{k+k'}\right)^2$$

$$T = \frac{\cancel{\text{const}}\ |C|^2 k'}{\cancel{\text{const}}\ |A|^2 k} = \frac{k'}{k}\left(\frac{2k}{k+k'}A\right)^2 \frac{1}{A^2}$$

$$\Rightarrow T = \frac{4kk'}{(k+k')^2}$$

b) every particle incident is either reflected or transmitted So $R+T=1$?

Check.

$$R+T = \frac{(k-k')^2}{(k+k')^2} + \frac{4kk'}{(k+k')^2} = \frac{k^2-2kk'+k'^2+4kk'}{(k+k')^2}$$

$$= \frac{k^2+2kk'+k'^2}{(k+k')^2} = \frac{(k+k')^2}{(k+k')^2} = 1$$

c) $\frac{1}{4} = \left(\frac{k-k'}{k+k'}\right)^2 \Rightarrow \frac{1}{2} = \frac{k-k'}{k+k'}$

$$\Rightarrow (k+k') = (k-k')\,2$$

$$\Rightarrow 3k' = k$$

$$\Rightarrow 3\sqrt{2m(E-U_0)/\hbar^2} = \sqrt{2mE/\hbar^2}$$

$$\Rightarrow 9\,\cancel{2m}(E-U_0)/\cancel{\hbar^2} = \cancel{2m}E/\cancel{\hbar^2}$$

$$\Rightarrow 8E = 9U_0 \Rightarrow E = \frac{9}{8}U_0$$

d) $T = \frac{4kk'}{(k+k')^2}$ → each contains a factor of $\sqrt{m}$

↳ cancel.

$(k+k')^2$ ↳ factor of $\sqrt{m}$ squared

No difference!

Note that this approach uses infinite plane wave solutions. These have a tricky interpretation in quantum theory. We could do the analysis using localized Gaussian wavepackets that evolve with time.

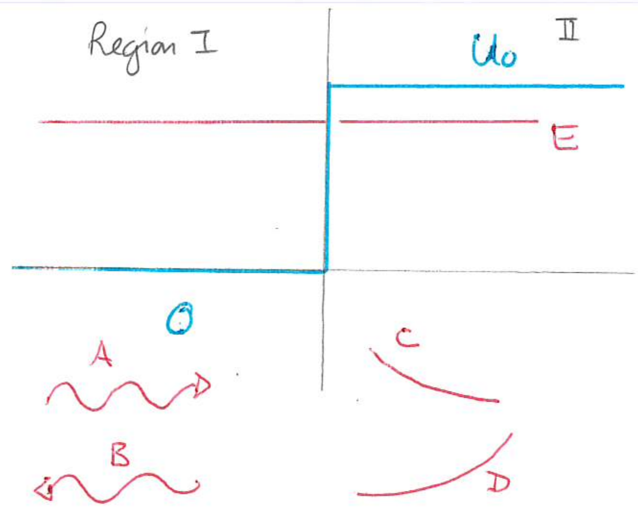
Demo: PhET Quantum Tunneling

* Step potential
* Gaussian - show evolution

Potential step: energy less than step

Consider the case where $E < U_0$ and a particle incident on the left. Classically there is no chance that the particle will be located in $x>0$.

We solve the TISE



Region I $(x<0)$

$$\psi_I = \underset{\text{incident}}{Ae^{ikx}} + \underset{\text{reflected}}{Be^{-ikx}} \quad \text{with} \quad k = \sqrt{2mE/\hbar^2}$$

Region II $(x>0)$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{II}}{dx^2} = (E-U_0)\psi_{II}$$

$$\Rightarrow \frac{d^2\psi_{II}}{dx^2} = \underbrace{2m\frac{U_0-E}{\hbar^2}}_{\text{positive}}\psi_{II}$$

Let

$$\alpha := \sqrt{2m(U_0 - E)/\hbar^2} > 0$$

Then

$$\frac{d^2\psi_{II}}{dx^2} = \alpha^2 \psi_{II}$$

which has solutions

$$\psi_{II} = Ce^{-\alpha x}$$

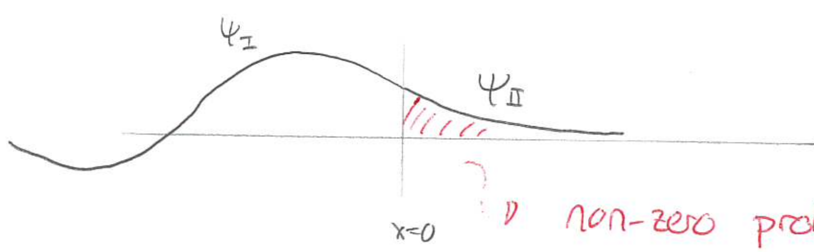
$$\psi_{II} = De^{+\alpha x}$$

The solution $De^{+\alpha x}$ cannot be normalized. Thus

$$\psi_{II} = Ce^{-\alpha x}$$

So we have:

Left of step	Right of step
$\psi_I = Ae^{ikx} + Be^{-ikx}$	$\psi_{II} = Ce^{\alpha x}$
$k = \sqrt{2mE/\hbar^2}$	$\alpha = \sqrt{2m(U_0 - E)/\hbar^2}$



$\Rightarrow$ non-zero probability of finding particle in right side!

Thus

$$\boxed{\text{There is a non-zero probability for locating the particle in a classically forbidden region.}}$$