Phys 231 2021

Lecture 27

Fri: Ex II Covers Lectures 13-26
HW 5-9

Bring: Previous 1/2 letter sheet
New 1/2 letter sheet.

Given Constants / Integrals

Review: 2020 Ex II Q1→7

Position + momentum measurements for stationary states

Stationary states satisfy the time independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\,\psi(x) = E\,\psi(x)$$

where E is a constant interpreted as energy. Then the time-dependent version of this is:

$$\Psi(x,t) = e^{-iEt/\hbar}\,\psi(x).$$

We can consider outcomes of measurements on such states. For example outcomes of position measurements are predicted by the probability density

$$P(x,t) = |\Psi(x,t)|^2$$

Quiz 1

Note that

$$P(x,t) = \Psi^*(x,t)\,\Psi(x,t)$$

and for stationary states:

$$P(x,t) = \left(e^{-iEt/\hbar}\,\psi(x)\right)^* e^{-iEt/\hbar}\,\psi(x)$$

$$= \cancel{e^{iEt/\hbar}}\,\psi^*(x)\,\cancel{e^{-iEt/\hbar}}\,\psi(x)$$

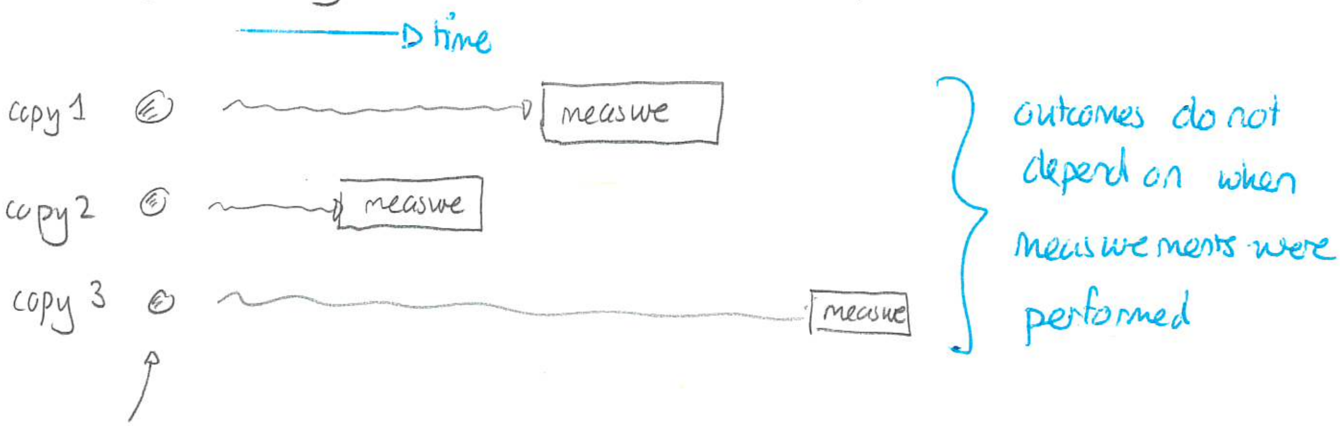
$$= \psi^*(x)\,\psi(x)$$

$$= |\psi(x)|^2$$

This is independent of time. Thus the statistics of position measurements do not depend on the time at which the measurement was made. For these reasons they are called stationary.

Quiz 2 — very few. — 20%

We can show that this is true for any type of measurement on any stationary state. The results do not depend on time



all in same stationary state.

Non-stationary states

It must be possible for quantum systems to evolve in such a way that measurement outcomes do depend on time. We ask whether this is possible and, if so, how.

Demo: PhET Quantum Bound

- harmonic oscillator
- create superposition
- observe wavefunction + prob density

The key mathematical fact is

> If $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are solutions to the Schrödinger equation then so is
>
> $$\Psi(x,t) = \alpha \Psi_1(x,t) + \beta \Psi_2(x,t)$$
>
> where α and β are any (complex) constants

We can form superpositions of infinitely many states.

Of special interest are superpositions of energy eigenstates. Suppose

$\psi_m(x)$ energy eigenstate $\rightsquigarrow$ energy E_m

$\psi_n(x)$ " " $\rightsquigarrow$ energy E_n

Then consider

$$\Psi(x,t) = \alpha \Psi_m(x,t) + \beta \Psi_n(x,t)$$

$$= \alpha e^{-iE_m t/\hbar} \psi_m(x) + \beta e^{-iE_n t/\hbar} \psi_n(x)$$

where $|\alpha|^2 + |\beta|^2 = 1$. This is such a superposition. We can ask about outcomes of measurements. For energy measurements the interpretation is:

	possible outcomes	probability
measure	E_m	$\lvert\alpha\rvert^2$
energy	E_n	$\lvert\beta\rvert^2$

We can determine probability densities. For position measurements:

$$P(x,t) = |\Psi(x,t)|^2$$

$$= \Psi^*(x,t)\Psi(x,t)$$

We consider the case where α, β are real.

1 Superpositions

Consider the superposition of the lowest two harmonic oscillator energy states:

$$\Psi(x,t) = \frac{1}{\sqrt{2}} e^{-iE_0 t/\hbar}\psi_0(x) + \frac{1}{\sqrt{2}} e^{-iE_1 t/\hbar}\psi_1(x).$$

Recall that $E_n = \hbar\omega_0(n + 1/2)$.

a) Determine an expression for the position probability density function.

b) Does the position probability density function depend on time?

c) Will the expectation value of the position oscillate? If so, with what frequency?

Answer:

a)

$$\Psi^* = \frac{1}{\sqrt{2}} e^{iE_0 t/\hbar}\psi_0^* + \frac{1}{\sqrt{2}} e^{iE_1 t/\hbar}\psi_1^*$$

$$\Psi^*\Psi = \left(\frac{1}{\sqrt{2}} e^{iE_0 t/\hbar}\psi_0^* + \frac{1}{\sqrt{2}} e^{iE_1 t/\hbar}\psi_1^*\right)\left(\frac{1}{\sqrt{2}} e^{-iE_0 t/\hbar}\psi_0 + \frac{1}{\sqrt{2}} e^{-iE_1 t/\hbar}\psi_1\right)$$

$$= \frac{1}{2}\left[\psi_0^*\psi_0 + e^{i(E_0-E_1)t/\hbar}\psi_0^*\psi_1 + e^{i(E_1-E_0)t/\hbar}\psi_1^*\psi_0 + \psi_1^*\psi_1\right]$$

$$= \frac{1}{2}\left\{|\psi_0|^2 + |\psi_1|^2 + e^{-i(E_1-E_0)t/\hbar}\psi_0^*\psi_1 + e^{i(E_1-E_0)t/\hbar}\psi_0\psi_1^*\right\}.$$

Now ψ_0, ψ_1 are real. So

$$P(x,t) = \frac{1}{2}\left\{|\psi_0|^2 + |\psi_1|^2 + \psi_0\psi_1\underbrace{\left(e^{-i(E_1-E_0)t/\hbar} + e^{+i(E_1-E_0)t/\hbar}\right)}_{2\cos[(E_1-E_0)t/\hbar]}\right\}$$

$E_1 = \frac{3}{2}\hbar\omega_0$

$E_0 = \frac{1}{2}\hbar\omega_0$

$$= \frac{1}{2}\left\{|\psi_0|^2 + |\psi_1|^2 + 2\psi_0\psi_1\cos(\hbar\omega_0 t/\hbar)\right\}$$

Thus

$$P(x,t) = \frac{1}{2}\left\{|\psi_0|^2 + |\psi_1|^2 + 2\psi_0\psi_1\cos(\omega_0 t)\right\}.$$

b) c) Yes oscillates with frequency ω_0