Phys 231 2021

Lecture 26

Tues: HW by 5pm

Weds: Read 5.9, 6.1

Fri: Exam II Covers Lectures 13 → 27
HW 5 → 9

2020 Exam II Q1 → Q7

Momentum measurements

Given many copies of a system with each in the same identical state, we can determine the momentum of each copy and then assemble the statistics of measurement outcomes. The expectation values of quantities associated with momentum can be determined using the momentum operator

$$\hat{p} : \text{function} \rightarrow \text{function}$$

$$\Psi(x,t) \rightarrow -i\hbar \frac{\partial \Psi(x,t)}{\partial x}$$

We will write

$$\hat{p}\,\blacksquare = -i\hbar \frac{\partial \blacksquare}{\partial x}$$ waiting for a function

Then we have that for a particle in state $\Psi(x,t)$ the expectation values are:

$$\langle p \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\, \hat{p}\, \Psi(x,t)\, dx$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \Psi(x,t)\, \hat{p}^2\, \Psi(x,t)\, dx \qquad \text{etc}, \ldots$$

Example: Consider a particle in an infinite well for $0 \le x \le L$.
Suppose that the particle is in a state for which at $t=0$

$$\Psi(x,t) = \begin{cases} \sqrt{\frac{30}{L^5}}\, x(x-L) & 0 \le x \le L \\ 0 & \text{otherwise.} \end{cases}$$

Determine a) $\langle p \rangle$

b) $\langle p^2 \rangle$

Answer: a)

$$\langle p \rangle = \int_{-\infty}^{\infty} \Psi(x)^* \left(-i\hbar \frac{\partial}{\partial x}\right) \Psi(x)\, dx$$

$$= -i\hbar \int_0^L \sqrt{\frac{30}{L^5}}\, x(x-L)\, \frac{\partial}{\partial x} \sqrt{\frac{30}{L^5}}\, x(x-L)\, dx$$

$$= -i\hbar\, \frac{30}{L^5} \int_0^L x(x-L)(2x-2xL)\, dx \qquad = 0$$

b)

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \Psi^*(x) \left(-i\hbar \frac{\partial}{\partial x}\right)\left(-i\hbar \frac{\partial}{\partial x}\right) \Psi(x)\, dx$$

$$= (-i\hbar)^2 \int_0^L \sqrt{\frac{30}{L^5}}\, x(x-L)\, \frac{\partial}{\partial x}\frac{\partial}{\partial x} \sqrt{\frac{30}{L^5}}\, x(x-L)\, dx$$

$$= -\hbar^2\, \frac{30}{L^5} \int_0^L x(x-L)\, \frac{\partial}{\partial x}(2x-L)\, dx$$

$$= -\hbar^2\, \frac{30}{L^5} \int_0^L x(x-L)\, 2\, dx$$

$$= -\frac{\hbar^2 60}{L^5} \int_0^L x(x-L)\, dx = -\frac{\hbar^2 60}{L^5} \int_0^L (x^2 - xL)\, dx$$

$$= -\frac{\hbar^2 60}{L^5} \left(\frac{x^3}{3} - \frac{x^2 L}{2}\right)\Bigg|_0^L = -\frac{\hbar^2 60}{L^5}\left(\frac{L^3}{3} - \frac{L^3}{2}\right) = \frac{\hbar^2 60}{L^5}\frac{L^3}{6}$$

$$\Rightarrow \langle p^2 \rangle = \frac{\hbar^2 10}{L^2}$$

So we have

Wavefunction for particle $\Psi(x,t)$

Expectation values of position measurements

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\, x\, \Psi(x,t)\, dx$$

$$\vdots$$

$$\langle x^n \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\, x^n\, \Psi(x,t)\, dx$$

Momentum operator

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

Expectation values of momentum measurements

$$\langle p \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t) \left(-i\hbar \frac{\partial}{\partial x}\right) \Psi(x,t)\, dx$$

$$= -i\hbar \int_{-\infty}^{\infty} \Psi^*(x,t) \frac{\partial \Psi}{\partial x}\, dx$$

$$\langle p^n \rangle = (-i\hbar)^n \int_{-\infty}^{\infty} \Psi^*(x,t) \frac{\partial^n \Psi}{\partial x^n}\, dx$$

Quiz 1

Energy expectation values

We can measure quantities that combine position and momentum and energy is an example. The measurement process might entail

copy 1 ⇝ $\boxed{\text{measure energy}}$ ⇝ outcome $E_{copy\ 1}$ e.g. ~~[illegible]~~ 3.0eV

copy 2 ⇝ $\boxed{\text{measure energy}}$ ⇝ outcome $E_{copy\ 2}$ e.g 2.0eV

⋮ e.g −3.0eV

each in state $\Psi(x,t)$ ⋮

what is average energy?

With a large number of copies we expect that the average energy will be the expectation value of the energy. If the energy has the form

$$E = \frac{1}{2m} p^2 + U(x)$$

then we can determine the expectation value by

$$\langle E \rangle = \frac{1}{2m} \langle p^2 \rangle + \langle U(x) \rangle$$

<u>Example:</u> Suppose that a particle in an infinite well $0 \le x \le L$ is in the state

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), & 0 \le x \le L \\ 0 & \text{otherwise} \end{cases}$$

This is the state associated with energy $E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$. Determine the expectation value of energy

<u>Answer:</u>

$$\langle E \rangle = \frac{1}{2m} \langle p^2 \rangle + \langle U(x) \rangle$$

$$= \frac{1}{2m} \int_{-\infty}^{\infty} \psi_n^*(x) (-i\hbar)^2 \frac{\partial^2}{\partial x^2} \psi_n(x)\, dx + \underbrace{\int_{-\infty}^{\infty} \psi_n^*(x)\, U(x)\, \psi(x)\, dx}_{U(x)\psi(x) = 0}$$

$$= -\frac{\hbar^2}{2m} \int_0^L \frac{2}{L} \sin\left(\frac{n\pi x}{L}\right) \frac{\partial^2}{\partial x^2} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{-\hbar^2}{mL} \int_0^L -\sin\left(\frac{n\pi x}{L}\right) \frac{n^2\pi^2}{L^2} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{n^2 \pi^2 \hbar^2}{mL^3} \underbrace{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx}_{L/2} \Rightarrow \langle E \rangle = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$

This is exactly the energy associated with the state. ∎

We can calculate $\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2}$ by evaluating $\langle E^2\rangle$. This uses (in the well)

$$E^2 = \frac{1}{4m^2} p^4$$

and we obtain

$$\langle E^2\rangle = \frac{1}{4m^2}\int_0^L \psi_n^*(x)\,(-i\hbar)^4 \frac{\partial^4}{\partial x^4}\psi_n(x)\,dx$$

$$= \frac{\hbar^4}{4m^2}\int_0^L \psi_n(x)\left(\frac{n\pi}{L}\right)^4 \psi_n(x)\,dx$$

$$= \frac{\hbar^4 n^4 \pi^4}{4mL^4} = \langle E\rangle^2$$

Thus $\Delta E = 0$. For this state, every energy measurement will yield exactly the same result

$$\left.\begin{array}{llll} \text{Copy 1} & \rightsquigarrow & \text{energy meas gives} & E_n = n^2\dfrac{\hbar^2\pi^2}{2mL^2} \\ \text{Copy 2} & \rightsquigarrow & \text{energy meas ''} & E_n = n^2\dfrac{\hbar^2\pi^2}{2mL^2} \\ \text{Copy 3} & \rightsquigarrow & \text{energy '' ''} & E_n = n^2\dfrac{\hbar^2\pi^2}{2mL^2} \end{array}\right\}$$

All exactly the same. Special feature of this state!

All in state $\psi_n(x)$

This is a feature of quantum theory. A more general treatment gives that, if a particle is in a stationary state / energy eigenstate with energy E_n then an energy measurement will yield outcome E_n with certainty.