Phys 231 2021

Lecture 24

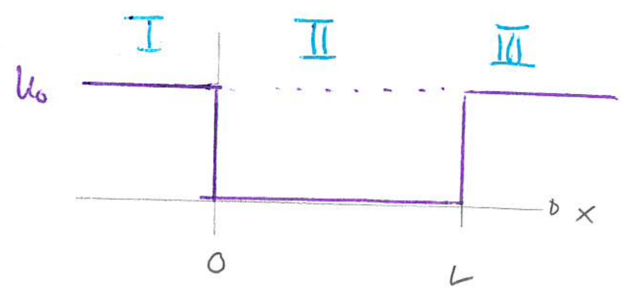
Thurs: Seminar

Fri: Read 5.7, 5.8, 5.9

Finite Well

A somewhat realistic model of a bound particle, is a particle trapped in a well with finite depth. Here

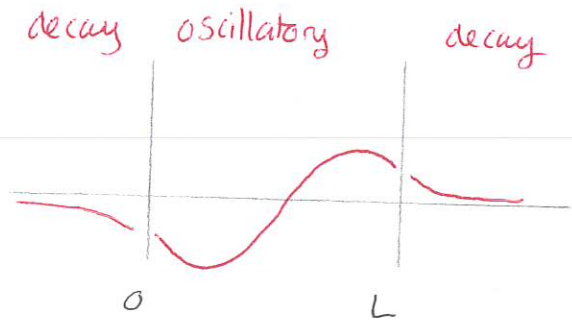
$$U(x) = \begin{cases} 0 & 0 \leq x \leq L \\ U_0 > 0 & \text{otherwise} \end{cases}$$



A bound energy eigenstate will be one for which $0 < E < U_0$. We seek such energies and the associated eigenstates. These will satisfy the TISE:

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi(x) = E\psi(x)$$

and one can form a qualitative picture based on the second derivative of ψ.



The strategy for solving the TISE is

1) obtain a solution in each region where U is constant
2) ensure that solutions are normalized. So as $x \to \pm\infty$ $\psi(x) \to 0$
3) match solutions at boundaries.

Region II Here $U=0$ and we denote the piece of the wave function by $\psi_{II}(x)$. Then

$$\frac{-\hbar^2}{2m}\frac{d^2\psi_{II}}{dx^2} = E\,\psi_{II} \quad \Rightarrow \quad \frac{d^2\psi_{II}}{dx^2} = -\frac{2mE}{\hbar^2}\psi_{II}$$

To solve this let

$$k = \sqrt{\frac{2mE}{\hbar^2}} \quad \Leftrightarrow \quad E = \frac{k^2\hbar^2}{2m}$$

Then

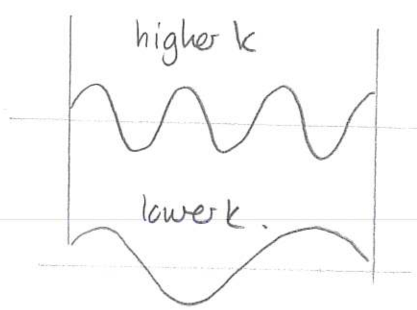
$$\frac{d^2\psi_{II}}{dx^2} = -k^2\psi_{II}$$

Quiz1 60%

Any linear combination of solutions is also a solution. Thus

$$\psi_{II}(x) = A\sin(kx) + B\cos(kx)$$

is the general solution in this region. Note that k determines how the solution oscillates. It also determines the energy.



Region I Here

$$\frac{-\hbar^2}{2m}\frac{d^2\psi_I}{dx^2} + U_0\psi_I = E\psi_I$$

$$\Rightarrow \quad \frac{d^2\psi_I}{dx^2} = \frac{U_0-E}{\hbar^2}2m\,\psi_I$$

Thus

$$\frac{d^2\Psi_I}{dx^2} = \alpha^2 \Psi_I \qquad \text{where} \qquad \alpha = \sqrt{\frac{2m(U_0-E)}{\hbar^2}}$$

The solutions to this are

$$\Psi_I = Ce^{\alpha x} + De^{-\alpha x}$$

where C, D are constants. The only way that this can be normalized is if $\Psi_I \to 0$ as $x \to -\infty$. Then as $x \to -\infty$

$$\Psi_I \to C \times 0 + D\infty \quad \Rightarrow D = 0$$

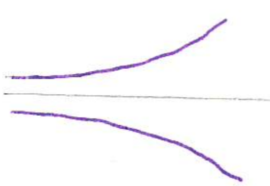
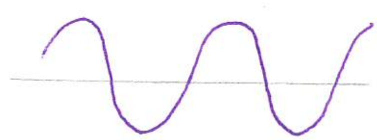
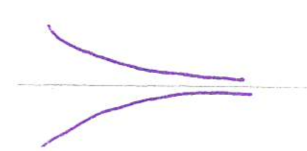
So the solution here is

$$\Psi_I(x) = Ce^{\alpha x}$$

Region III Similar to region I but as $x \to \infty$ $\Psi_{III} \to 0$ gives

$$\Psi_{III}(x) = De^{-\alpha x}$$

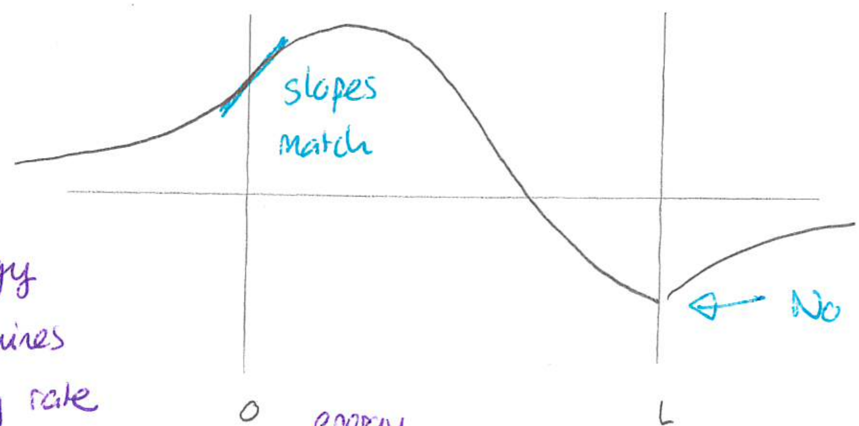
So we get solutions in each region: (so far valid for any $E < U_0$)

I	II	III
$\Psi_I(x) = Ce^{\alpha x}$	$\Psi_{II}(x) = A\sin(kx) + B\cos(kx)$	$\Psi_{III}(x) = De^{-\alpha x}$
		

We must match functions and first derivatives at the boundary.

Why do these constrain the energies?

Suppose we guess an energy $E < U_0$. This fixes α and k. So it fixes the decay rate and the rate of oscillation. Starting in region I we plot. $\Psi_I(x)$



energy determines decay rate

energy determines oscillation rate, k and where oscillating function "starts"

← No way to match here

⇓

This energy is not valid.

So we find that only certain very special energies allow for all the matching to occur. These are the possible energy values and they will differ from the results for an infinite well.

Demo: PhET Band States

- show wave function + prob.

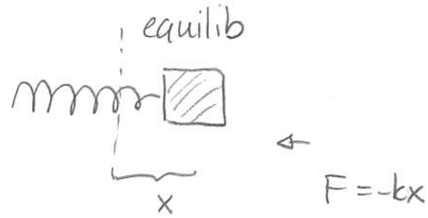
Quantum dots

The finite well serves as a model for a quantum dot which is a particle trapped within a certain region.

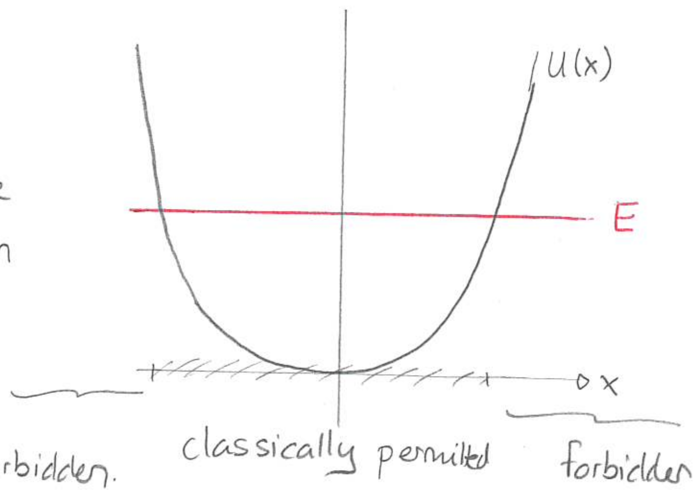
Demo: Research Gate QD article

Harmonic Oscillator

A harmonic oscillator is a system in which there is a single force that tends to restore the system state to equilibrium.
The force must be proportional to the displacement from equilibrium. Then the potential has the form



$$U = \frac{1}{2} k x^2$$



where $k > 0$ is a constant. Again such systems in classical physics are bound by the constraint $E > U$. In general such systems will oscillate. For example if the particle has mass m then the angular frequency of oscillation is

$$\omega_0 = 2\pi f = \sqrt{\frac{k}{m}} \Rightarrow m\omega_0^2 = k$$

Thus an alternative expression for the potential is:

$$U = \frac{1}{2} m \omega_0^2 x^2$$

Such potentials apply to particles on a microscopic scale and the physics of these is described by quantum theory. Examples of this include

1) trapped ions
2) nuclei in crystal lattices
3) standing waves in optical systems / lasers
4) vibrations of molecules. Demo: Molecular Normal Modes