Mon: Read 5.1, 5.2

Tues: HW by 5pm

Schrödinger Equation

We have a mechanism for describing how to use wavefunctions to predict outcomes of measurements. We now need a mechanism for determining

- which wavefunctions are relevant in various situations?
- given a wavefunction that describes a physical situation at one instant, what wavefunction describes the situation at a later instant?

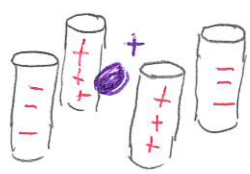
Examples of such situations are:

1) free particles

no external influences

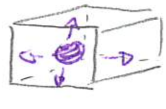
$\Psi(x,t) = ??$

2) trapped ion (restricted to one dimension)



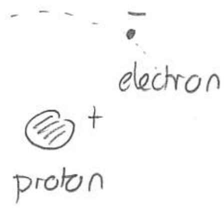
Demo: U Innsbruck Trapped Ions

3) trapped particle (in otherwise free region)



Demo: Quantum Dot Articles

4) electron in a hydrogen atom



$\Psi(x,y,z,t) = ??$

We need a general prescription for:

Specify wavefunction at initial instant $t=0$

$\Psi(x,0)$??

$\longrightarrow$ "Propagate" wavefunction to later instants

$\Psi(x,0) \xrightarrow{??} \Psi(x,t)$

$\hookrightarrow$ Determine statistics for outcome of any measurement

The general procedure will involve a differential equation that is constructed via:

Describe the system in terms of total energy e.g.

$$E = \underset{\text{kinetic}}{K} + \underset{\text{potential}}{U}$$

$\longrightarrow$ Use energy to construct an equation that the wavefunction must satisfy at all times and locations.

$\hookrightarrow$ wavelike differential equation
$\equiv$ Schrödinger Equation

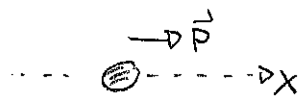
We do not have a way to derive the Schrödinger equation from any existing theory. We simply make an informed guess about the equation and then check the consequences.

Schrödinger equation for a free particle in one dimension

Consider a free particle that can move along the x-axis and which has a well defined momentum, p.

First consider the classical description of this particle's state. It would involve momentum $\vec{p}$ and position x and a well-defined energy E.

Classically



Quiz 1   50% - 60%

Now consider what is already known in terms of wavefunctions. We know that

$$\Psi(x,t) = Ae^{i(px/\hbar - \omega t)}$$

works well to describe particle interference in slit experiments. We now need an assumption for ω. Recall that for a single photon

$$E = \hbar\omega \Rightarrow \omega = E/\hbar .$$

So we assume $\omega = E/\hbar$ and then

$$\Psi(x,t) = Ae^{i(px - Et)/\hbar}$$

What equation does this satisfy? Consider time evolution. Usually this involves a derivative w.r.t time. So

$$\frac{\partial \Psi}{\partial t} = Ae^{i(px-Et)/\hbar}\frac{\partial}{\partial t}\left(ipx/\hbar - \frac{iEt}{\hbar}\right)$$

$$= Ae^{i(px-Et)/\hbar}\left(\frac{-iE}{\hbar}\right) = \frac{-i}{\hbar}E\Psi$$

$$\Rightarrow \quad i\hbar\frac{\partial \Psi}{\partial t} = E\Psi$$

Quiz 2

Consider $$\frac{\partial \Psi}{\partial x} = \frac{\partial}{\partial x} A e^{(ipx/\hbar - iEt/\hbar)}$$

$$= \frac{ip}{\hbar} A e^{i(px/\hbar - Et/\hbar)}$$

$$= \frac{ip}{\hbar} \Psi$$

Then $$\frac{\partial^2 \Psi}{\partial x^2} = \frac{ip}{\hbar} \frac{\partial \Psi}{\partial x} = \frac{ip}{\hbar}\left(\frac{ip}{\hbar}\Psi\right) = -\frac{p^2}{\hbar^2}\Psi$$

So $$\frac{1}{2m}\frac{\partial^2 \Psi}{\partial x^2} = \frac{-p^2}{2m\hbar^2}\Psi$$

$$= -\frac{1}{\hbar^2} E\Psi \implies E\Psi = \frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}$$

But we have $E\Psi = i\hbar \frac{\partial \Psi}{\partial t}$ which gives:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}$$

Thus we have:

If for a free particle

$$\Psi(x,t) = A e^{i(px - Et/\hbar)}$$

then

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}$$

This covers one special case for a free particle wave function. However, we assume that:

For any free particle the wavefunction $\Psi(x,t)$ satisfies

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} = i\hbar \frac{\partial \Psi}{\partial t}$$

This is the time-dependent Schrödinger equation for a free particle. The task now is to solve this equation. By a solution we mean:

$\Psi(x,t)$ is a solution if, upon substitution into the Schrödinger equation, it is true for all x and t.

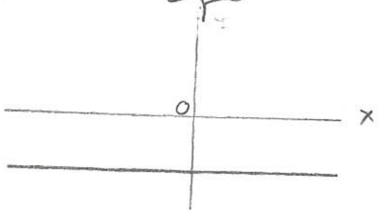
Quiz 3

Left side:

$$\frac{\partial \Psi}{\partial x} = 2A(x-Bt)$$

$$\frac{\partial^2 \Psi}{\partial x^2} = 2A$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} = -\frac{\hbar^2}{m}A$$

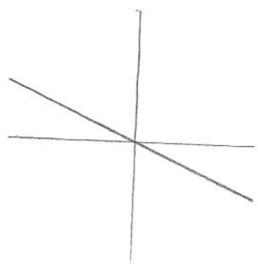


Right side

$$\frac{\partial \Psi}{\partial t} = 2A(x-Bt)(-B)$$

$$= -2AB(x-Bt)$$

$$i\hbar\frac{\partial \Psi}{\partial t} = -2iAB(x-Bt)$$



at $t=0$

Imag. part

Not the same