Phys 231 2021

Lecture 19

Thurs: Seminar

Fri: Read 4.3

Momentum Wavefunction

The statistics of outcomes of measurements are described using:

Position measurements	Momentum measurements
position wavefunction $\Psi(x,t)$	momentum wavefunction $\tilde{\Psi}(p,t)$
$\downarrow$	$\downarrow$
Prob density $P(x,t) = \lvert\Psi(x,t)\rvert^2$	Prob density $\tilde{P}(p,t) = \lvert\tilde{\Psi}(p,t)\rvert^2$

The two wavefunctions are related by

$$\boxed{\tilde{\Psi}(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-i(px/\hbar - \omega t)}\, \Psi(x,t)\, dx}$$

Here ω is yet to be specified. We can ignore this by considering the wavefunctions at $t=0$

1 Position and Momentum Uncertainty for Gaussian Wavefunctions

The position wavefunction for a particle at $t = 0$ happens to be

$$\psi(x) = \frac{1}{\sqrt{a}(2\pi)^{1/4}} e^{-x^2/4a^2}$$

where $a > 0$ is a constant with units of meters. The expectation value and uncertainty for position measurements for a collection of particles in this state are:

$$\langle x \rangle = 0$$
$$\Delta x = a.$$

The momentum wavefunction at $t = 0$ is

$$\tilde{\psi}(p) = \sqrt{\frac{2a}{\hbar}} \frac{1}{(2\pi)^{1/4}} e^{-p^2a^2/\hbar^2}$$

The following exercise uses the integrals

$$\int_{-\infty}^{\infty} e^{-(\alpha x^2+\beta x+\gamma)} \mathrm{d}x = \sqrt{\frac{\pi}{\alpha}} e^{\beta^2/4\alpha} e^{-\gamma}$$
$$\int_{-\infty}^{\infty} x e^{-\alpha x^2} \mathrm{d}x = 0$$
$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{2\alpha^{3/2}}.$$

provided that $\alpha > 0$. These are valid for any β and γ which could be complex.

a) Determine the expectation value of momentum $\langle p \rangle$.

b) Determine the uncertainty in the momentum Δp.

c) Consider various Gaussian wavefunctions. As the position of the particle can be specified more precisely, what happens to the momentum of the particle?

d) Determine an expression that relates the uncertainties in the position and momentum.

Answer: a) $\tilde{P}(p) = |\tilde{\psi}(p)|^2 = \frac{2a}{\hbar} \frac{1}{\sqrt{2\pi}} e^{-p^2 2a^2/\hbar^2}$

$$\langle p \rangle = \int_{-\infty}^{\infty} p \tilde{P}(p) dp = \frac{2a}{\hbar} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p e^{-p^2 2a^2/\hbar^2} = 0$$

$\Rightarrow \langle p \rangle = 0$

b) $$\Delta p = \sqrt{\langle p^2 \rangle - \underbrace{\cancel{\langle p \rangle^2}}_{0}}$$

$$= \sqrt{\langle p^2 \rangle}$$

Then
$$\langle p^2 \rangle = \int_{-\infty}^{\infty} p^2 \tilde{P}(p)\, dp$$

$$= \frac{2a}{\hbar} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2 2a^2/\hbar^2}\, dp$$

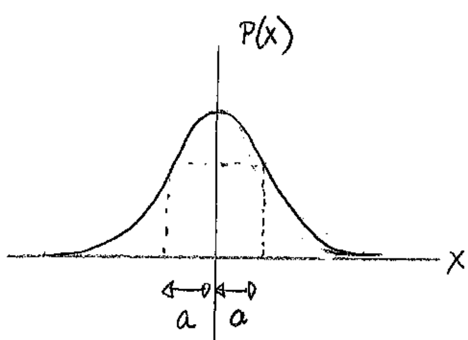
$$= \frac{2a}{\hbar} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2 \alpha}\, dp \qquad \text{with } \alpha = 2a^2/\hbar^2$$

$$= \frac{\cancel{2}a}{\hbar} \frac{1}{\sqrt{2\cancel{\pi}}} \frac{\sqrt{\cancel{\pi}}}{\cancel{2}(2a^2/\hbar^2)^{3/2}}$$

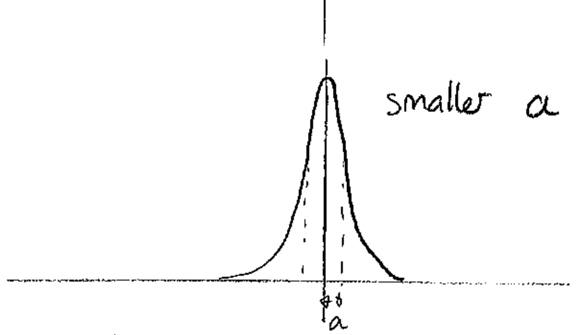
$$= \frac{1}{\hbar} \frac{a}{2^{1/2}\, 2^{3/2}} \frac{\hbar^3}{a^3} = \frac{\hbar^2}{4a^2}$$

Thus $$\boxed{\Delta p = \frac{\hbar}{2a}}$$

c)



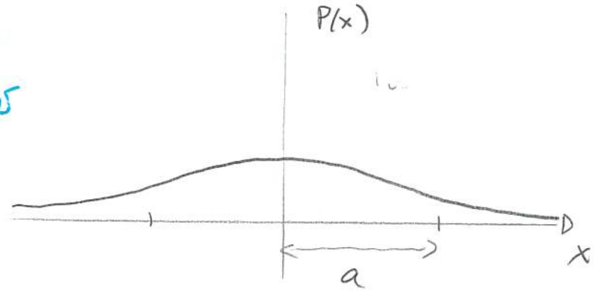
The width of $P(x)$ is quantified by $\Delta x = a$. Decreasing a means that one specifies the position more precisely



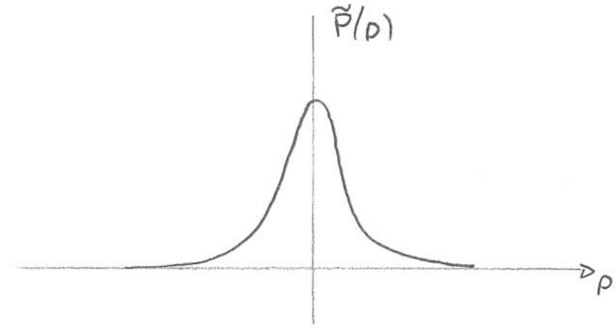
smaller $a \Rightarrow$ narrower range of likely positions

Now consider the momentum. This is also centered at $p=0$ but with width inversely related to a.

~~larger~~ ~~so~~ larger a

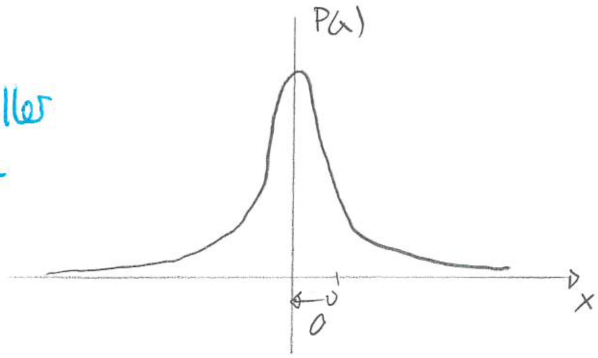


less precise position

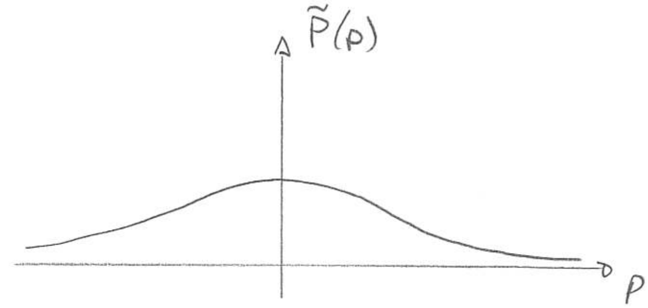


more precise momentum

smaller a



more precise position



less precise momentum.

So as the position is specified more precisely, the momentum is specified less precisely and vice-versa.

d) $\Delta x = a$

$$\Delta p = \frac{\hbar}{2a} = \frac{\hbar}{2\Delta x} \Rightarrow \Delta p \Delta x = \frac{\hbar}{2}$$

This quantifies the trade-off in information about position and momentum

Uncertainty principle

We have seen that for particles described by a Gaussian wavefunction, there is a trade-off between information about position and information about momentum. Perhaps there are other wavefunctions which constrain the position and momentum simultaneously and more precisely than the Gaussian wavefunction. It turns out that this is impossible. A mathematical theorem gives:

> For any wavefunction, the uncertainties in position and momentum satisfy
>
> $$\Delta x \Delta p \geq \hbar/2$$
>
> where Δx = uncertainty in position
>
> Δp = " " momentum

This is the Heisenberg uncertainty principle. In general it stipulates that there is a trade off in position and momentum.

Quiz 1 30% → 100%

Quiz 2 90%

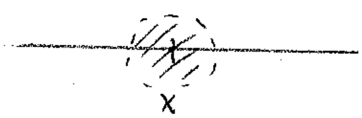
Quiz3 100%

The Heisenberg uncertainty principle informs us that, regardless of how much we try we cannot specify both the position and momentum of a particle in one dimension with arbitrary precision. There will always be some statistical fluctuations in measurement outcomes and this

Quantum theory is inherently statistical

<u>Example</u> (This is a one dimensional analysis of a more complicated three dimensional situation).

Consider a neutron located within a nucleus (approximate width 10^{-14}m). We are tempted to say that the nucleus location is roughly $x=0$ and it is at rest with speed $v=0$.



How accurately can we say this?

<u>Answer:</u> We know $\Delta x \leq 10^{-15}$m and

$$\Delta p \Delta x \geqslant \frac{\hbar}{2} \Rightarrow \Delta p \geqslant \frac{\hbar}{2\Delta x}$$

$$\geqslant \frac{\hbar}{2\times10^{-14}\,\text{m}} = \frac{6.63\times10^{-34}\,\text{J.s}}{4\pi\times10^{-14}\,\text{m}}$$

$$\Rightarrow \Delta p \geqslant 5.3\times10^{-19}\ \text{kgm/s}$$

$$m\Delta v \geqslant 5.3\times10^{-19}\ \text{kgm/s}$$

$$1.67\times10^{-27}\,\text{kg}\ \Delta v \geqslant 5.3\times10^{-19}\ \text{kgm/s}$$

$$\Delta v \geqslant 3\times10^{6}\ \text{m/s}$$ large range.