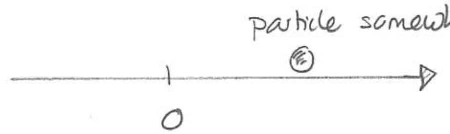
Tues: HW 5pm
Weds: Read 4.4, 4.7

Wavefunctions: position variables

The wave function for a particle in one dimension is usual constructed with regard to the position variable, x. Thus we have



$\rightsquigarrow$ state of particle described by wave function $\Psi(x,t)$

The wavefunction constructed gives information about the outcome of position measurements via the probability density

$$P(x,t) = |\Psi(x,t)|^2$$

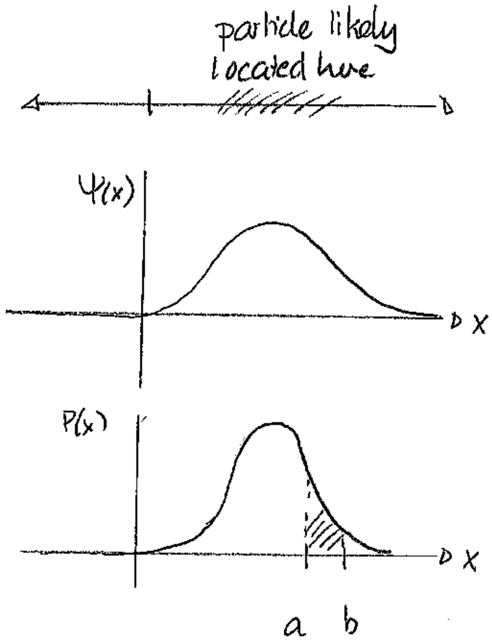
This will give complete statistical information about any outcomes of position measurements. There is no more complete information available for the particle given that this particular wavefunction describes its state. The particular types of information that one can extract are:

* probabilities of measurement outcomes

$$\text{Prob}(a \leq x \leq b) = \int_a^b P(x,t)\,dx = \int_a^b |\Psi(x,t)|^2\,dx$$

* expectation values, uncertainties of measurement outcomes
* correlations between measurement outcomes, ...

So for example at one instant



wavefunction

$$\Psi(x) = \frac{1}{\sqrt{a}(2\pi)^{1/4}} e^{-(x-x_0)^2/4a^2}$$

(is real – can graph

$\downarrow$

$$P(x) = |\Psi(x)|^2$$

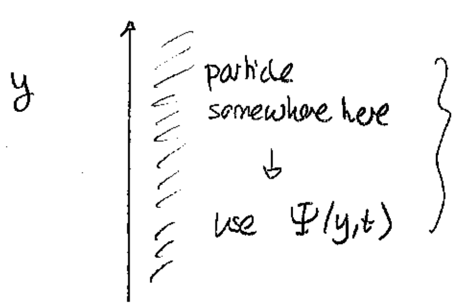
$$= \frac{1}{a\sqrt{2\pi}} e^{-(x-x_0)^2/2a^2}$$

<u>measure position</u>

$$\text{Prob}(a \leq x \leq b) = \int_a^b \frac{1}{a\sqrt{2\pi}} e^{-(x-x_0)^2/2a^2} dx. \quad = ??$$

$$= \text{area between } a, b$$

We can use the wavefunction to construct probabilities and provide statistics for measurement outcomes for any quantity that only depends on position (plus some constant parameters). For example if we consider a particle that moves vertically where the position variable is y then we would describe its state by a wavefunction $\Psi(y,t)$ which gives the most complete information about vertical position. But we could also use this to provide information about



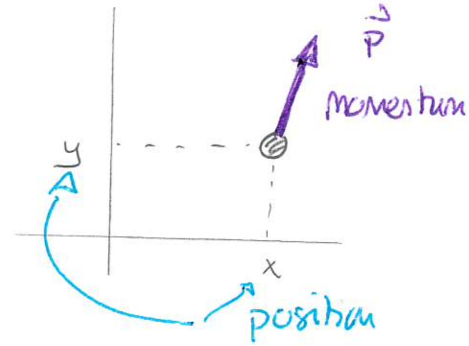
$\rightarrow$ Prob density

$P(y,t) = |\Psi(y,t)|^2$

gravitational potential energy, $U = mgy$, and address any question about the measurement of gravitational potential energy.

Momentum measurements

In classical physics we can describe the state of any particle by both its position and momentum. We know that in quantum physics anything to do with position measurements must be described using the wavefunction that describes the state of the particle $\Psi(x,t)$. How can we describe what happens when we do momentum measurements?



It will emerge that we cannot use the position wavefunction directly. We cannot hope to somehow use $\Psi(x,t)$ at two different times to get a velocity and then multiply by mass. The way that this will be done is to create a momentum wavefunction and use this to describe outcomes of momentum measurements. Thus

> The state of a particle constrained to move in one dimension, x, can be described by a momentum wavefunction
>
> $$\tilde{\Psi}(p,t)$$
>
> where p is the momentum variable. The probability density for momentum measurements is
>
> $$\tilde{P}(p,t) = |\tilde{\Psi}(p,t)|^2$$

We then get the usual statistical results:

Prob(momentum meas gives p in range $p_1 \to p_2$)

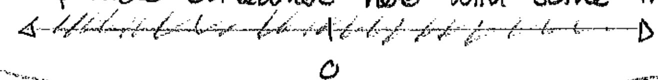
$$= \int_{p_1}^{p_2} \tilde{P}(p,t)\,dp = \int_{p_1}^{p_2} |\tilde{\Psi}(p,t)|^2\,dp$$

Expectation value of momentum is

$$\langle p \rangle = \int_{-\infty}^{\infty} p\,\tilde{P}(p,t)\,dp = \int_{-\infty}^{\infty} p\,|\tilde{\Psi}(p,t)|^2\,dp$$

and so on. It now appears that we have two ways to describe the state of a particle

particle somewhere here with some momentum



Position wavefunction

$\Psi(x,t)$

describes position measurements

Momentum wave function

$\tilde{\Psi}(p,t)$

describes momentum measurements

These must be related. Based on various other considerations the connection is:

The momentum wavefunction is obtained from the position wavefunction by

$$\tilde{\Psi}(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-i(px/\hbar - \omega t)}\, \Psi(x,t)\,dx$$

1 Momentum wavefunction

The position wavefunction for a particle at $t = 0$ happens to be

$$\psi(x) = \frac{1}{\sqrt{a}(2\pi)^{1/4}} e^{-x^2/4a^2}$$

where $a > 0$ is a constant with units of meters. The expectation value and uncertainty for position measurements for a collection of particles in this state are:

$$\langle x \rangle = 0$$
$$\Delta x = a.$$

The following exercise uses the integrals

$$\int_{-\infty}^{\infty} e^{-(\alpha x^2 + \beta x + \gamma)} \mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/4\alpha}\, e^{-\gamma}$$
$$\int_{-\infty}^{\infty} x e^{-\alpha x^2} \mathrm{d}x = 0$$
$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} \mathrm{d}x = \cancel{\frac{\sqrt{\pi}}{2\alpha^{3/2}}}. \quad \frac{\sqrt{\pi}}{2\alpha^{3/2}}$$

provided that $\alpha > 0$. These are valid for any β and γ which could be complex.

a) Determine an expression for the momentum wavefunction, $\tilde{\psi}(p)$.

b) Determine an expression for the probability density, $\tilde{P}(p)$ for outcomes of momentum measurements.

c) Determine the expectation value and uncertainty for momentum measurement outcomes.

Answer: a)

$$\tilde{\Psi}(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-i(px/\hbar - \omega t)}\, \Psi(x,t)\, dx$$

At $t = 0$

$$\tilde{\Psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \frac{1}{\sqrt{a}\,(2\pi)^{1/4}} e^{-x^2/4a^2}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\sqrt{a}\,(2\pi)^{1/4}} \int_{-\infty}^{\infty} e^{-\left(x^2/4a^2 + \frac{ip}{\hbar} x\right)} dx$$

Here $\alpha = \dfrac{1}{4a^2}$ $\quad \beta = \dfrac{ip}{\hbar}$ $\quad \gamma = 0$

So

$$\tilde{\Psi}(p) = \frac{1}{\sqrt{2\cancel{\pi}\hbar}} \frac{1}{\sqrt{a}\,(2\pi)^{1/4}} \sqrt{\frac{\pi}{1/4a^2}}\; e^{\left(\frac{ip}{\hbar}\right)^2 / \frac{1}{a^2}}$$

$$= \frac{1}{\sqrt{2\hbar}} \frac{1}{(2\pi)^{1/4}} \frac{2a}{\sqrt{a}}\; e^{-p^2a^2/\hbar^2}$$

$$\tilde{\Psi}(p) = \frac{1}{(2\pi)^{1/4}} \sqrt{\frac{2a}{\hbar}}\; e^{-p^2a^2/\hbar^2}$$

b) $$\tilde{P}(p) = |\tilde{\Psi}(p)|^2 = \frac{1}{\sqrt{2\pi}} \frac{2a}{\hbar}\; e^{-p^2a^2 2/\hbar^2}$$

c) $$\langle p \rangle = \int_{-\infty}^{\infty} p\, \tilde{P}(p)\, dp$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2a}{\hbar} \int_{-\infty}^{\infty} p\, e^{-p^2a^2 2/\hbar^2} dp = 0$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\langle p^2 \rangle}$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} p^2 \tilde{P}(p)\, dp = \frac{1}{\sqrt{2\pi}} \frac{2a}{\hbar} \int_{-\infty}^{\infty} p^2 e^{-p^2a^2 2/\hbar^2} dp$$

$$= \frac{1}{\sqrt{2\cancel{\pi}}} \frac{\cancel{2}a}{\hbar} \frac{\sqrt{\cancel{\pi}}}{\cancel{2}\,(a^2 2/\hbar^2)^{3/2}}$$

$$= \frac{a}{\sqrt{2}\,\hbar} \frac{\hbar^3}{a^3(2^{3/2})} = \frac{\hbar^2}{4a^2}$$

$$\Rightarrow \Delta p = \frac{\hbar}{2a}$$