Phys 231 2021

Lecture 17

Mon: Read 4.3, 4.4

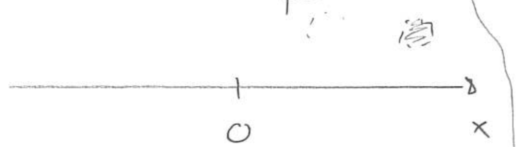
Tues: HW by 5pm

Wavefunctions for particles in one dimension

Recall the scheme for describing particles in one dimension

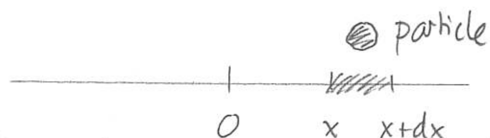
PHYSICAL PICTURE

particle

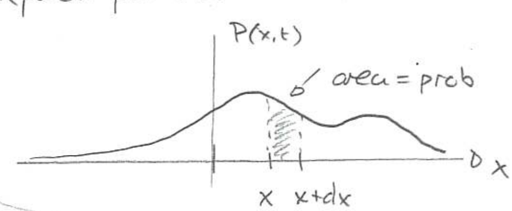


measure position

Get an outcome in range x to $x+dx$



Repeat many times on identically prepared particles



predicts this

MATH / CALCULATIONS

Associate complex wavefunction with particle

$$\Psi(x,t)$$

Probability density for position measurement outcomes

$$P(x,t) = \underbrace{|\Psi(x,t)|^2}_{\text{modulus square} \equiv \text{real}}$$

Probability of outcome in range $x \to x+dx$

$$\text{Prob}(x \to x+dx) = P(x,t)\, dx$$

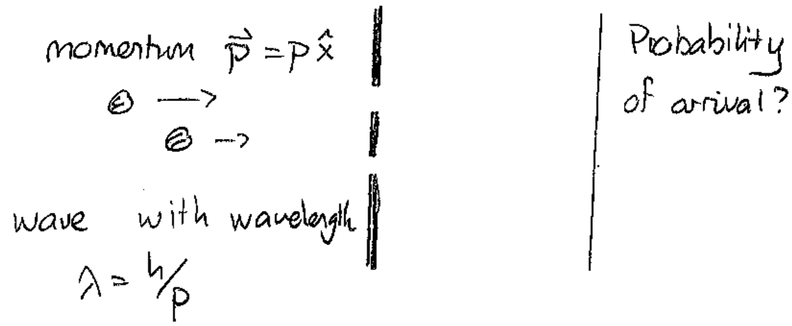
We now consider candidate wavefunctions for various situations.

Complex exponential /sinusoidal wavefunctions.

Recall that we could explain particle interference experiments by associating a wave with the particles.

For particles with momentum p the relevant wavelength is

$$\lambda = h/p$$



Then the form of the wave that explained these was a two dimensional version of the complex exponential

$$\Psi(x,t) = A e^{i(kx-\omega t)}$$

Note that $k = 2\pi/\lambda = 2\pi/(h/p) = p\frac{2\pi}{h} = \frac{p}{\hbar}$. Thus a possibly useful wavefunction for such particles at least before they reach the barrier is

$$\boxed{\Psi(x,t) = A e^{i(px/\hbar - \omega t)}}$$

Note that for such a situation

1) the particle travels in the +x direction
2) the particle has a precise momentum, p.

Quiz 1 - 0% → 10%

Quiz 2 → 80%

In these cases the probability density for position measurement outcomes is

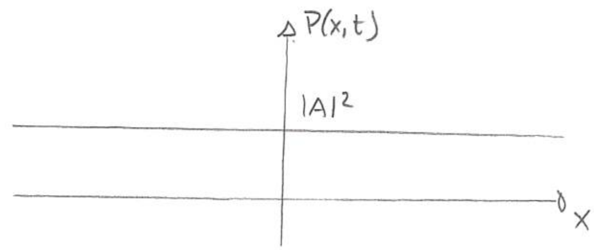
$$P(x,t) = |\Psi(x,t)|^2$$

$$= \left| A e^{i(px/\hbar - \omega t)} \right|^2$$

$$= |A|^2 \underbrace{\left| e^{i(px/\hbar - \omega t)} \right|^2}_{=1}$$

$$\Rightarrow \quad P(x,t) = |A|^2$$

This gives a uniform probability density over all positions. It also gives a probability density that does not depend on time.



However, one issue with this is that the probability density cannot be normalized. We require, that at any time

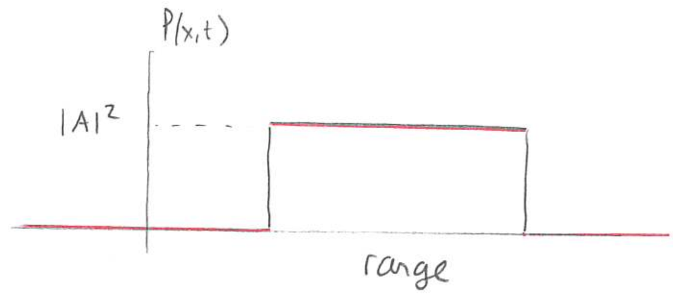
$$\int_{-\infty}^{\infty} P(x,t)\, dx = 1 \quad \Rightarrow \quad \int_{-\infty}^{\infty} |A|^2 dx = 1 \quad \Rightarrow \quad |A|^2 \underbrace{\int_{-\infty}^{\infty} dx}_{\infty} = 1$$

and this is impossible.

We clearly cannot use such a wavefunction to predict probabilities unless we restricted its range, such as

$$\Psi(x,t) = \begin{cases} A e^{i(px/\hbar - \omega t)} & \text{some finite range} \\ 0 & \text{everywhere else} \end{cases}$$

In this case the probability density is still independent of t. However, there will be other issues with this



* there will not be a single definite momentum
* it may not satisfy the correct equation as demaned by physics, particularly at the edges.

Gaussian wavefunctions

We will clearly need wavefunctions whose magnitudes decrease appreciably at infinite distances. There are various possibilites, which we consider at a single instant in time. One common possibility is a Gaussian wavefunction

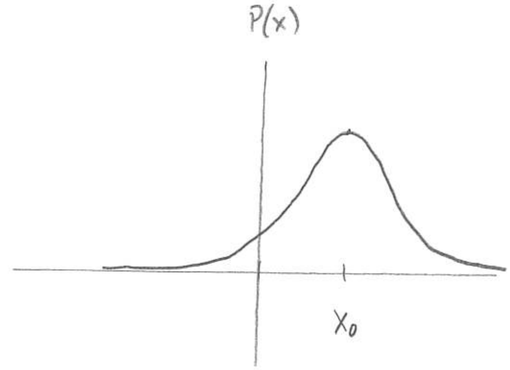
$$\Psi(x) = A e^{-(x-x_0)^2/4a^2}$$

Quiz 3 70%-

Here

$$P(x) = |\Psi(x)|^2$$

$$= A^2 \, e^{-(x-x_0)^2/2a^2}$$



gives a probability distribution concentrated around $x = x_0$. We can show that

1) normalization requires $\quad A = \dfrac{1}{\sqrt{a}\,(2\pi)^{1/4}}$

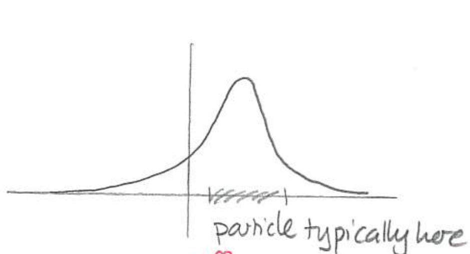
2) the expectation value of the position measurement outcome is

$$\langle x \rangle = x_0$$

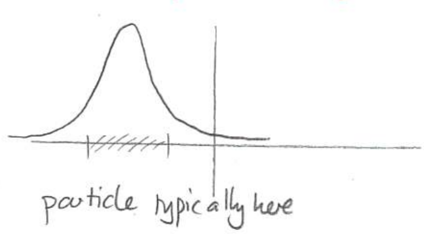
3) the uncertainty (standard deviation) in position measurements is

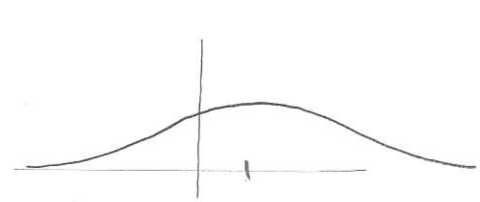
$$\Delta x \equiv \sigma_x = a$$

Thus such Gaussian wavefunctions can be used to describe situations where a particle is localized with both the locality and the spread in outcomes included.

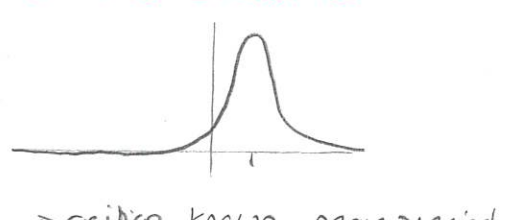


same a, different x_0





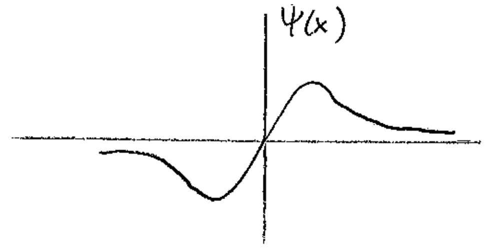
same x_0 different a



Other localized wavefunctions

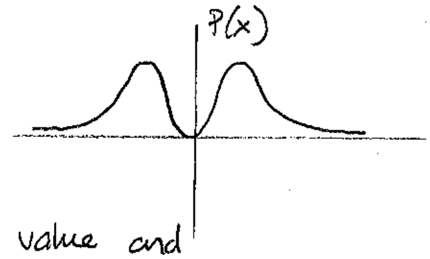
Other localized wavefunctions are possible. For example we can have products of Gaussian wavefunctions and banded functions. An example might be

$$\Psi(x) = A\, x\, e^{-x^2/4a^2}$$



Then

$$P(x) = A^2 x^2 e^{-x^2/2a^2}$$



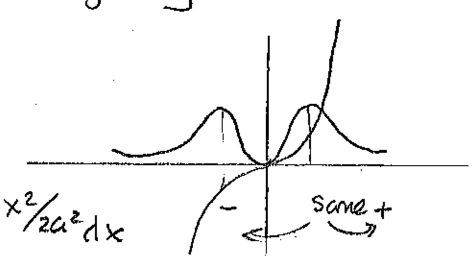
<u>Example:</u> Determine the normalization, expectation value and standard deviation for this.

<u>Answer:</u> Need $\int P(x)\,dx = 1 \Rightarrow A^2 \underbrace{\int_{-\infty}^{\infty} x^2 e^{-x^2/2a^2} dx}_{\sqrt{2\pi}\, a^3} = 1$

$$\Rightarrow A^2 \sqrt{2\pi}\, a^3 = 1 \qquad \Rightarrow A = \frac{1}{(2\pi)^{1/4} a^{3/2}}$$

Then $\langle x \rangle = \int_{-\infty}^{\infty} x P(x)\,dx = A^2 \int \underbrace{x^3}_{\text{anti}} \underbrace{e^{-x^2/2a^2}}_{\text{symmetry}} dx$

$$= 0$$



Finally $\langle x^2 \rangle = \int x^2 P(x)\,dx = A^2 \int_{-\infty}^{\infty} x^4 e^{-x^2/2a^2} dx$

$$= 3a^2$$

$$\Rightarrow \Delta x = \sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{3}\, a \Rightarrow \Delta x = \sqrt{3}\, a$$