Weds: HW by 5pm

Thurs: Seminar 12:30

- My usual Zoom link.

Wavefunctions for particles in one dimension

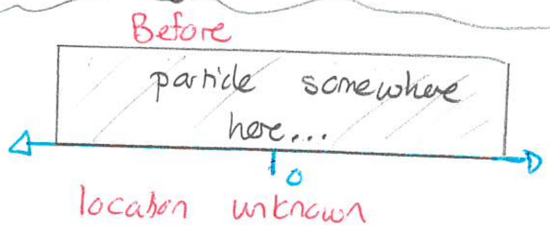
A variety of physical phenomena involving particles could not easily be described using conventional classical physics. These could be described by associating a wave with the particles. Examples are:

1) particle interference + diffraction experiments with multiple slits
2) particle scattering experiments
3) Bohr model.

We want a general scheme that uses waves to describe a wide variety of situations involving particles. The initial description will focus on the outcome of position measurements or, equivalently, the particle location. We also initially consider the one-dimensional case. We assume

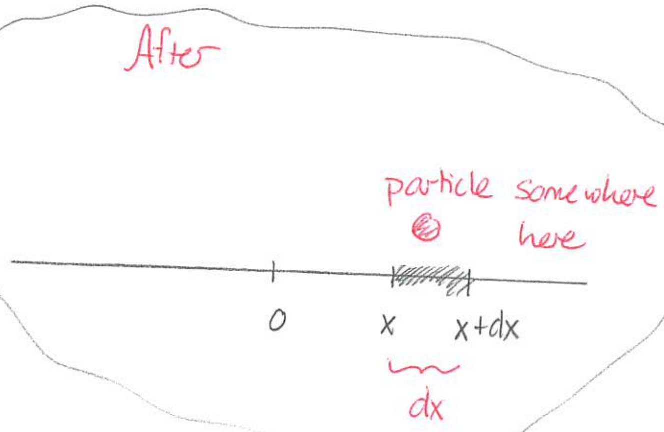
> The outcome of measuring the position of the particle will be a location (or small range of locations) along the x-axis

Schematically



Measuring device invoked and measurement performed

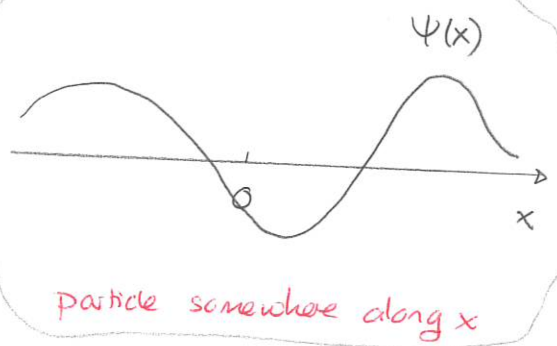
predict?? quantum theory!!



Describe the measurement outcome as x (with resolution dx)

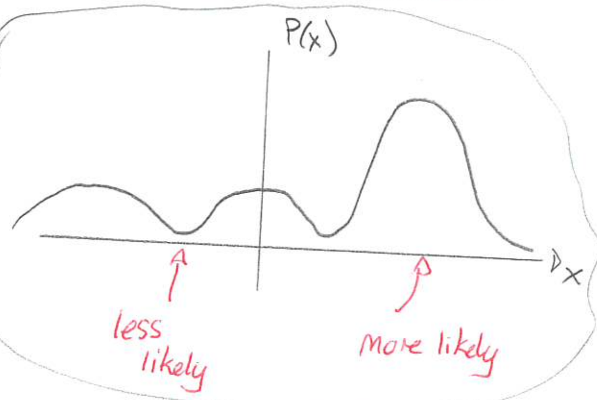
What quantum theory will do is to predict the probability with which the outcome occurs. The procedure for doing this is:

At any given instant in time and based on the physical situation, associate a complex wave function $\Psi(x)$ with the particle



Probability density for position measurement outcomes at this instant is

$$P(x) = |\Psi(x)|^2$$



Thus we get:

If the wavefunction associated with the particle is $\psi(x)$ then the probability with which a position measurement yields an outcome in the range $x \to x+dx$ is

$$P(x)\,dx = |\psi(x)|^2 dx$$

In general the state of the particle will change with time. Thus the position probability density and the wavefunction must vary with time.

The state of a particle whose location is restricted to points along the x-axis is described by a complex wavefunction $\Psi(x,t)$, which typically depends on time. The probability density for position measurement outcomes at time t is

$$P(x,t) = \underbrace{|\Psi(x,t)|^2}_{\text{modulus squared} \Rightarrow \text{real}}$$

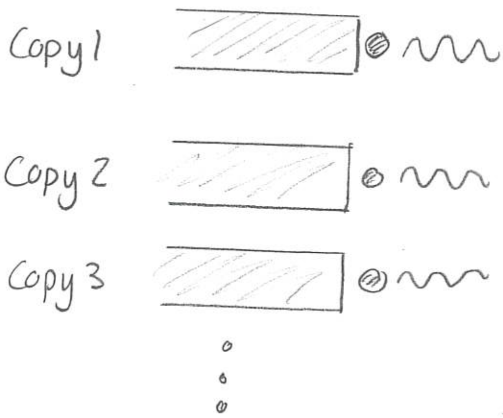
Quiz 1 40% same

In general a measurement of position will yield a single outcome within a small range, dx (and we can in principle make this infinitesimally small).

Quiz 2

What the wavefunction does is describe the statistics of repeated measurements on the same situation. We could imagine many identical copies of particles "prepared" in the same way

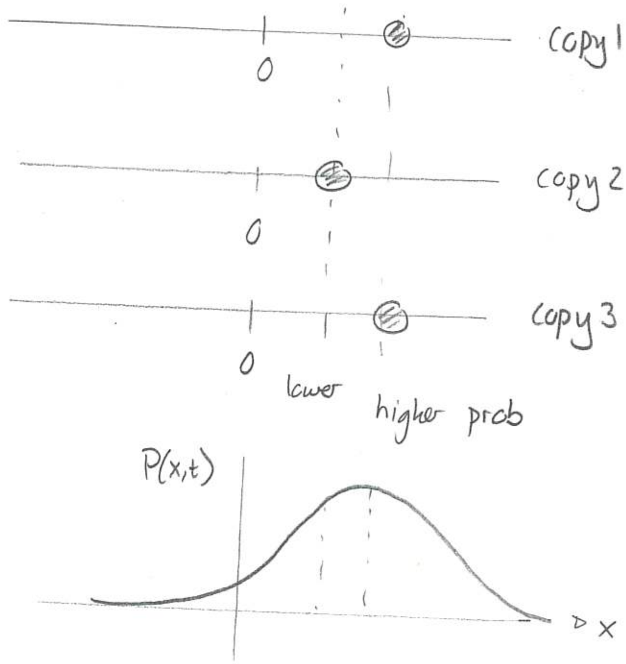
At time $t=0$



each do exactly the same thing

$\Rightarrow$ same $\Psi(x,0)$

measure position at a later time



quantum theory predicts this

$$P(x,t) = |\Psi(x,t)|^2$$

Quiz 3

Note that the wavefunction at any single moment does not describe the past or subsequent state of the particle. It does not, without the time part give any information about the evolution of the state of the particle. This will require an equation that does describe how the particle subsequently evolves.

We now consider the process of constructing plausible wavefunctions.

Once we have the wavefunction we can:

* determine the probabilities with which position measurement outcomes occur
* determine the mean an expectation values of position measurement outcomes $\langle x \rangle$ and σ_x
* determine the statistics of the outcomes of any variable that only depends on x.

Example: In a particular situation the wavefunction for a particle is at $t=0$

$$\Psi(x,0) = \begin{cases} A e^{i\alpha} \sqrt{x}\, e^{-x/2b} & x>0 \\ 0 & \text{otherwise} \end{cases}$$

where $b>0$ and $\alpha>0$ are constants.

a) Determine the probability density for position measurement outcomes. Determine A

b) Determine the probability with which the position measurement yields an outcome in the range $0 < x < b$

c) Determine the expectation value of position measurements and the uncertainty

d) Determine the most probable position measurement outcome

Note:

$$\int x e^{ax} = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$$

$$\int x^2 e^{ax} = \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right)$$

$$\int x^4 e^{ax} dx = \frac{e^{ax}}{a}\left[x^4 - \frac{4x^3}{a} + \frac{12x^2}{a^2} - \frac{24x^4}{a^3} + \frac{24}{a^4}\right]$$

Answer: a) $P(x,0) = |\Psi(x,0)|^2$

$$= \begin{cases} A^2 \underbrace{|e^{i\alpha}|^2}_{1} \, x e^{-x/b} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

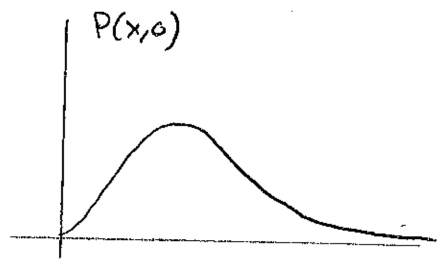
$$\int_0^\infty P(x,0)dx = 1$$

$$\Rightarrow \quad A^2 \int_0^\infty x \, e^{-x/b} dx = 1$$

$$\Rightarrow \quad A^2 \left[-b e^{-bx} (x+b) \right]_0^\infty = 1$$

$$\Rightarrow \quad A^2 b^2 = 1 \quad \Rightarrow \quad A = 1/b$$

Thus $P(x,0) = \begin{cases} \frac{1}{b^2} x e^{-x/b} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$



b) $$\int_0^b P(x,0)dx = \frac{1}{b^2} \int_0^b x \, e^{-x/b} dx$$

$$= 1 - 2/e = 0.264$$

c) $\langle x \rangle = \int_{-\infty}^{\infty} x\, P(x,0)\, dx$

$$= \frac{1}{b^2}\int_0^{\infty} x^2 e^{-x/b} dx = 2b$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

Then

$$\langle x^2 \rangle = \int_0^{\infty} x^2 P(x,0)\, dx = \frac{1}{b^2}\int_0^{\infty} x^3 e^{-x/b} dx = 6b^2$$

So $\sigma_x = \sqrt{6b^2 - 4b^2} = \sqrt{2}\, b$

d) Need max of $P(x)$.

$$\frac{dP}{dx} = 0 \Rightarrow \frac{d}{dx}\left(x e^{-x/b}\right) = 0$$

$$\Rightarrow e^{-x/b} + x\left(-\frac{1}{b}\right) e^{-x/b} = 0$$

$$\Rightarrow 1 - x/b = 0 \Rightarrow x = b$$