Phys 231 2021

Lecture 15

Weds: HW by 5pm

- note problem that requires random numbers.

Weds: Read 4.3

## Probabilities: Continuous Outcomes

Suppose the events/outcomes are labeled by a continuous variable, $x$. Then the probability density satisfies:

$$\boxed{\begin{array}{l} 0 \leq P(x) \\ \int_{-\infty}^{\infty} P(x)\,dx = 1 \end{array}}$$

— positive

— normalized

The mean, or expectation value of the distribution is:

$$\boxed{\bar{x} \equiv \langle x \rangle = \int_{-\infty}^{\infty} x\,P(x)\,dx}$$

In many cases this quantifies the typical value returned when the distribution is sampled. The range of values returned is quantified via the standard deviation

$$\boxed{\sigma_x = \sqrt{(\overline{x^2}) - (\bar{x})^2}}$$

This requires calculation of

$$\overline{x^2} = \int_{-\infty}^{\infty} x^2 P(x)\,dx$$

Example: Consider the distribution:

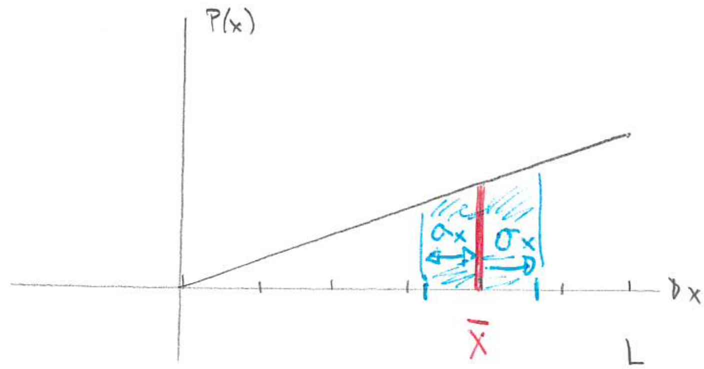
$$P(x) = \begin{cases} 0 & x \le 0 \\ \frac{2}{L^2}x & 0 \le x \le L \\ 0 & L \le x \end{cases}$$

Determine:

a) mean (expectation value)

b) standard deviation

Answer a) $\bar{x} = \int_{-\infty}^{\infty} x\, P(x)\, dx$




$$\bar{x} = \int_0^L x\left(\frac{2}{L^2}x\right)dx$$

$$= \frac{2}{L^2}\int_0^L x^2\, dx$$

$$= \frac{2}{L^2}\,\frac{x^3}{3}\Big|_0^L \quad \Rightarrow \quad \bar{x} = \frac{2}{3}L$$

b) use $\sigma_x = \sqrt{\overline{x^2} - (\bar{x})^2}$ Then

$$\overline{x^2} = \int_{-\infty}^{\infty} x^2\, P(x)\, dx = \int_0^L x^2 \frac{2}{L^2} x\, dx = \frac{2}{L^2}\int_0^L x^3 dx$$

$$\overline{x^2} = \frac{1}{2L^2} L^4 = \frac{L^2}{2}$$

Thus $\sigma_x = \sqrt{\frac{L^2}{2} - \left(\frac{2}{3}L\right)^2} = \sqrt{L^2\left(\frac{1}{2} - \frac{4}{9}\right)} = L\frac{1}{\sqrt{18}}$

$$\Rightarrow \sigma_x = \frac{1}{\sqrt{2}}\,\frac{L}{3}$$

# Gaussian or normal distribution

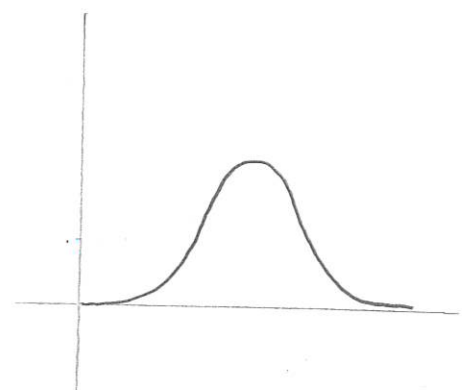
We often want to consider situations where the likely outcomes are concentrated in a narrow region amongst all the possibilities.

Demo: Plinko (PhET)

-use 26 Rows -run - show distribution

We note that the distribution is peaked around the midpoint (for equal left/right probability). If we tilt the left right probability the peak shifts. The particular distribution in this case is called the binomial distribution.

In the limit as the number of levels approaches $\infty$ the distribution can be approximated by a Gaussian, or normal distribution:



$$P(x) = B\, e^{-(x-x_0)^2/2a^2}$$

Quiz 1 40%

We can fix the constant B by normalizing. We need

$$\int_{-\infty}^{\infty} P(x)\, dx = 1 \Rightarrow B \int_{-\infty}^{\infty} e^{-(x-x_0)^2/2a^2} dx = 1$$

Then integrals of this type can be evaluated.

Slide

In general, we have that:

$$\int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \qquad \int_{-\infty}^\infty e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^\infty e^{-(\alpha x^2 + \beta x + \gamma)} dx = \sqrt{\frac{\pi}{\alpha}}\, e^{-(\beta^2 - 4\alpha\gamma)/4\alpha}$$

In this situation we use $u = x - x_0 \Rightarrow du = dx$

$$\Rightarrow B \int_{-\infty}^\infty e^{-u^2/2a^2} du = 1$$

$$\Rightarrow B \sqrt{\frac{\pi}{1/2a^2}} = 1 \quad \Rightarrow \quad B\sqrt{2\pi a^2} = 1$$

Assuming $a > 0$ we get $B = \frac{1}{\sqrt{2\pi}} \frac{1}{a}$

Thus a general form of the normal distribution is:

$$\boxed{P(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{a} e^{-(x-x_0)^2/2a^2}}$$

We can compute the mean of this distribution. Then

$$\bar{x} = \int_{-\infty}^\infty x P(x) dx = \frac{1}{\sqrt{2\pi}} \frac{1}{a} \underbrace{\int_{-\infty}^\infty x e^{-(x-x_0)^2/2a^2} dx}_{\text{look up / MAPLE}}$$

$$= x_0$$

Thus we have $P(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{a} e^{-(x-\bar{x})^2/2a^2}$

Quiz 2 — 10%

The width of the distribution is determined by $a$ alone

Slide 2,3

We can show that

$$\boxed{\text{The distribution } P(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{a} e^{-(x-\bar{x})^2/2a^2} \text{ the standard deviation is } \sigma = a}$$

Proof: We have $\sigma = \sqrt{\overline{x^2} - \bar{x}^2}$ and

$$\overline{x^2} = \int_{-\infty}^{\infty} x^2 P(x)\,dx = \frac{1}{\sqrt{2\pi}\,a}\underbrace{\int_{-\infty}^{\infty} x^2 e^{-(x-\bar{x})^2/2a^2}\,dx}$$

$$= \frac{1}{\sqrt{2\pi}\,a}\sqrt{2\pi}\,a\,(a^2+\bar{x}^2) = a^2+\bar{x}^2$$

Then $\overline{x^2} - \bar{x}^2 = a^2 \Rightarrow \sigma = a$ ∎

The normal distribution is important in probability theory in general because of the central limit theorem. This says that if:

1) one samples a distribution and calculates the sample average and standard deviation of the mean, then

2) the sample average will be distributed according to a normal distribution with true mean equal to the mean of the original distribution and with standard deviation equal to the standard deviation of the mean of the original distribution