Phys 231 2021

Lecture 14

Fri: ~~[illegible]~~ Exam I

Covers Lectures 1 → 12

HW 1 → 4

Bring – ½ Letter Sheet single side

– calculator

Given Constants

Review: 2020 Exam I

Probabilities: Discrete Outcomes

The general framework for all discrete outcome situations is:

Events / Outcomes	Associated quantities, q	Probabilities
$\{i\}$	$\{q_i\}$	$\{p_i\}$

and the probabilities must satisfy

$$0 \le p_i \le 1$$

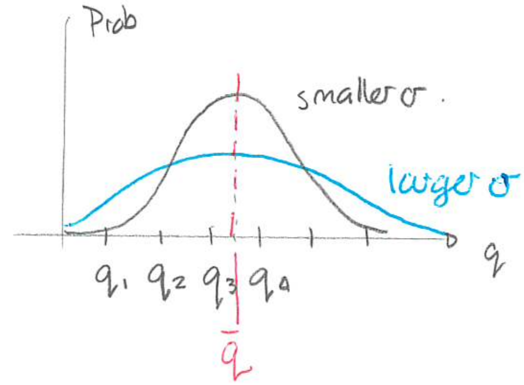
$$\sum_{\text{all } i} p_i = 1$$

Then the mean of q is $\bar{q} := \sum p_i q_i$

The standard deviation is defined as:

$$\boxed{\sigma = \sqrt{\sum_{\text{all } i} (q_i - \bar{q})^2 p_i}}$$

For many types of probability distribution the mean quantifies the central most likely outcomes/values and the standard deviation quantifies the spread.



A convenient way to determine the standard deviation is to use:

$$\sigma = \sqrt{\overline{(q^2)} - (\bar{q})^2}$$

Example: The two level Plinko game has outcomes A, B, C. and payouts, q, as illustrated. Determine the standard deviation of this:

outcome	q	p
A	12	$1/4$
B	6	$1/2$
C	12	$1/4$

Answer: $p_A = 1/4$

$p_B = 1/2$

$p_C = 1/4$

Thus $\bar{q} = p_A q_A + p_B q_B + p_C q_C = 9$. Now we need the mean of the square:

$$\overline{q^2} = \sum_i p_i q_i^2 = p_A q_A^2 + p_B q_B^2 + p_C q_C^2 = \frac{1}{4}144 + \frac{1}{2}36 + \frac{1}{4}144 = 36 + 18 + 36$$
$$= 90$$

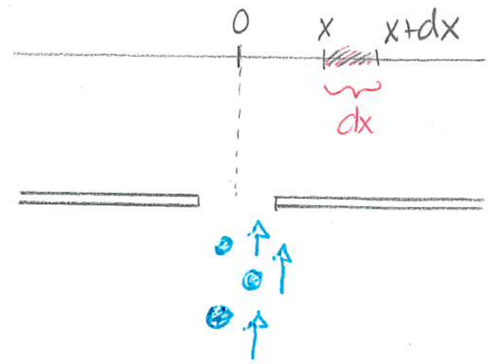
So $\sigma = \sqrt{90 - 9^2} = \sqrt{9} \Rightarrow \sigma = 3$

This has the meaning that if we play the game N times then the average payout per ball will be very likely to be in the range

$$\bar{q} - \sigma/\sqrt{N} \to \bar{q} + \frac{\sigma}{\sqrt{N}} \quad \equiv \quad 9 - \frac{3}{\sqrt{N}} \to 9 + \frac{3}{\sqrt{N}}$$

average payout.

Probability: Continuous Outcomes



Suppose that we fire particles toward a barrier and aim to predict the horizontal location of arrival on a distant screen. There are infinitely (non-countable) locations at which any particle could arrive and we could never describe the probability with which a particle will arrive in exactly one location. Rather we can ask about the probability of arrival within a range of locations:

$$x \to x + dx$$

There will be a countable number of such locations and we can assign a probability to each such "bin". This will as $dx \to 0$ be proportional to dx. So

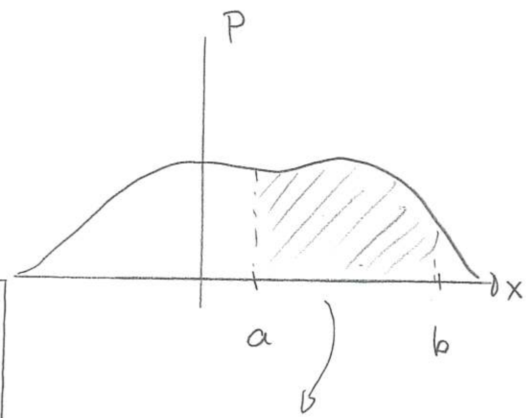
$$\boxed{\text{Prob}(\text{location is in range } x \to x+dx) = P(x)dx}$$

where $P(x)$ is the probability density. We can immediately see that there is a constraint:

$$\boxed{0 \leq P(x)}$$

More precisely the meaning of this is given in terms of an extended range.

So

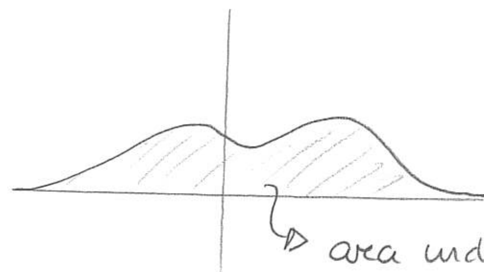


$$\boxed{\text{Prob(event is in } a \leq x \leq b) = \int_a^b P(x)\,dx}$$

Prob = area under $P(x)$

Since any outcome must occur we have an additional requirement on the probability density:

$$\boxed{\int_{-\infty}^{\infty} P(x)\,dx = 1}$$



area under entire curve = 1.

This is the <u>normalization condition</u>

Quiz 1 10% - 100%

In this example

$$P(x) = \begin{cases} \alpha x & 0 \leq x \leq L \\ 0 & \text{otherwise} \end{cases}$$

Normalization implies:

$$\int_{-\infty}^{\infty} P(x)\,dx = 1 \quad \Rightarrow \quad \int_0^L \alpha x\,dx = 1$$

$$\Rightarrow \quad \frac{\alpha x^2}{2}\Big|_0^L = 1 \quad \Rightarrow \quad \frac{\alpha L^2}{2} = 1 \quad \Rightarrow \quad \alpha = \frac{2}{L^2}$$

Thus $P(x) = \begin{cases} \frac{2}{L^2} x & 0 \leq x \leq L \\ 0 & \text{otherwise} \end{cases}$

Note that $P(x)$ has units of m^{-1}.

Quiz 2 80%

For continuous distributions like this:

The mean of x is:

$$\bar{x} = \int_{-\infty}^{\infty} \underbrace{x}_{\text{outcome}} \underbrace{P(x)\,dx}_{\text{prob}} \equiv \langle x \rangle \sim \sum \overset{\text{outcome}}{q_i}\,\overset{\text{prob}}{p_i}$$

expectation value of x

The standard deviation of x is defined via:

$$\boxed{\sigma_x^2 := \int_{-\infty}^{\infty} (x-\bar{x})^2 P(x)\,dx}$$

We can show that

$$\sigma_x = \sqrt{\overline{(x^2)} - (\bar{x})^2}$$

$$= \sqrt{\langle x^2 \rangle - (\langle x \rangle)^2}$$

Example: Consider the distribution:

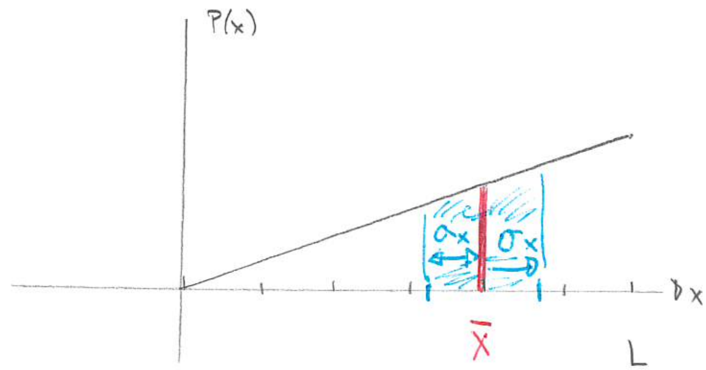
$$P(x) = \begin{cases} 0 & x \leq 0 \\ \frac{2}{L^2}x & 0 \leq x \leq L \\ 0 & L \leq x \end{cases}$$

Determine:

a) mean (expectation value)

b) standard deviation

Answer a) $\bar{x} = \int_{-\infty}^{\infty} x\, P(x)\, dx$



$$\bar{x} = \int_0^L x\left(\frac{2}{L^2}x\right)dx$$

$$= \frac{2}{L^2}\int_0^L x^2\, dx$$

$$= \frac{2}{L^2}\,\frac{x^3}{3}\Big|_0^L \quad \Rightarrow \quad \bar{x} = \frac{2}{3}L$$

b) use $\sigma_x = \sqrt{\overline{x^2} - (\bar{x})^2}$ Then

$$\overline{x^2} = \int_{-\infty}^{\infty} x^2\, P(x)\, dx = \int_0^L x^2\,\frac{2}{L^2}x\, dx = \frac{2}{L^2}\int_0^L x^3\, dx$$

$$\overline{x^2} = \frac{1}{2L^2}L^4 = \frac{L^2}{2}$$

Thus $\sigma_x = \sqrt{\frac{L^2}{2} - \left(\frac{2}{3}L\right)^2} = \sqrt{L^2\left(\frac{1}{2} - \frac{4}{9}\right)} = L\frac{1}{\sqrt{18}}$

$$\Rightarrow \quad \sigma_x = \frac{1}{\sqrt{2}}\,\frac{L}{3}$$