Mon: HW due by 5pm

Weds: Read

Fri: Exam I - Covers Lectures 1 → 12

HW 1 → 4

Note on products of complex numbers

We we form a product of complex numbers, the real parts cannot be extracted just from the individual real parts. Consider

$$z = \underbrace{e^{i\alpha}}_{z_1} \underbrace{3e^{i\pi/4}}_{z_2}$$

Then $z_1 = \cos\alpha + i\sin\alpha \rightarrow$ Real part $= \cos\alpha$

$z_2 = 3\cos\pi/4 + 3i\sin\pi/4 \rightarrow$ " " $= 3\cos\pi/4$

But the real part of z is not the product $3\cos\alpha\cos\pi/4$, because when one multiplies the two imaginary parts both contribute a real number. Rather

$$z = z_1 z_2 = (\cos\alpha + i\sin\alpha)(3\cos\pi/4 + 3i\sin\pi/4)$$

$$= 3\cos\alpha\cos\pi/4 - 3\sin\alpha\sin\pi/4 + i\left[\sin\alpha\, 3\cos\pi/4 + 3\sin\pi/4\cos\alpha\right]$$

$$= \underbrace{3\cos(\alpha+\pi/4)} + i\, 3\sin(\alpha+\pi/4)$$

Real part of $z = 3\cos(\alpha+\pi/4)$

More rapidly $\quad z = 3e^{i(\alpha+\pi/4)} = \underbrace{3\cos(\alpha+\pi/4)} + 3i\sin(\alpha+\pi/4)$

real part $= 3\cos(\alpha+\pi/4)$

Probability

Quantum theory will usually predict the probabilities of outcomes of various events. The necessary branch of mathematics that describes this is called probability theory. This provides a mathematical framework for describing situations where events due not occur with certainty.

Probability for discrete outcomes

The simplest formulations of probability theory involve events whose outcomes are discrete. Examples are:

1) coin toss – yields one of heads or tails (2 possible outcomes)
2) die roll – " " " 1,2,3,4,5,6 (six possible outcomes)

In each case we cannot predict the outcome of any single "run" or "trial" with certainty. However, we can often predict the fraction of times that any outcome can occur if we repeat the "run" many times. Probability theory describes this via:

List all possible events $\{i\}$	List probability p_i with which event occurs
$i=1$	p_1
$i=2$	p_2
$i=3$	p_3
$\vdots$	$\vdots$

e.g. ~~die~~ roll

die roll

i	p_i
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

coin toss

i	p_i
H	1/2
T	1/2

The intuitive idea behind this is that if we do many trials, say N in total, then if N_i is the number of times i occurs,

$$p_i \approx \frac{N_i}{N}$$

This then provides two constraints:

$$\boxed{\begin{array}{c} 0 \le p_i \le 1 \\ \displaystyle\sum_{\text{all events } i} p_i = 1 \end{array}}$$

The set of probabilities that satisfy this, one for each event, is called a <u>probability distribution</u>.

When we construct probability distributions we have to specify the set of events and associated probabilities carefully.

Quiz1 80% -100%

Quiz2 90% -

Note that we can create probabilities for events, each of which contains multiple outcomes, from those with individual outcomes. If the individual outcomes are independent then we multiply the probabilities for the individual outcomes to determine those for the multiple outcomes.

<u>Quiz3</u> 70% – 90%

<u>Demo</u>: PhET Plinko Probability – Lab Tab
– Run with many – show ideal

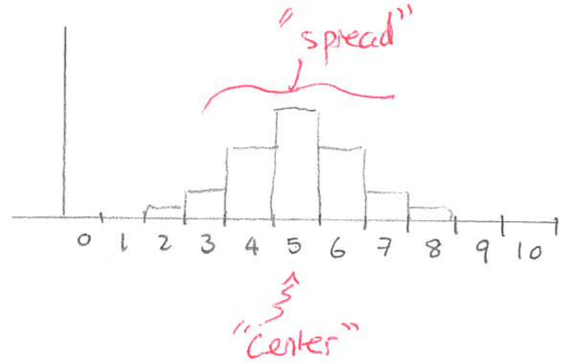
<u>Mean and standard deviation</u>

A complete description of a probability distribution requires the probability for each outcome. However, we can attain some useful information about the distribution based on

- the "center" of the distribution
- the "spread" " "   "

<u>Demo</u>: PhET Plinko Prob – Use many rows

For many distributions the "center" is quantified by the mean and this requires that each outcome be quantified



event	associated value	probability
1	q_1	p_1
2	q_2	p_2
3	q_3	p_3
4	q_4	p_4
5	q_5	$\vdots$
$\vdots$	$\vdots$	
i	q_i	p_i

Then the mean of q is

$$\boxed{\bar{q} := \sum_{\text{events } i} p_i q_i}$$

Quiz 4 -90%

In this example:

event	payout (q)	prob (p)
A	12	$\frac{1}{4}$
B	4	$\frac{1}{2}$
C	12	$\frac{1}{4}$

$$\Rightarrow \quad \bar{q} = 12 \cdot \frac{1}{4} + 4 \frac{1}{2} + 12 \cdot \frac{1}{4}$$

$$= 3 + 4 + 3 \quad = 8$$

This would represent the typical payout per run since after N runs the total payout is:

$$N_A \cdot q_A + N_B q_B + N_C \cdot q_C = q_{tot}$$

and the payout per run is (as $N \to \infty$)

$$\frac{q_{tot}}{N} = \frac{N_A}{N} q_A + \frac{N_B}{N} q_B + \frac{N_C}{N} q_C = p_A q_A + p_B q_B + p_C q_C = \sum p_i q_i = \bar{p}$$

We can extend this to any quantity constructed from q. For example, suppose

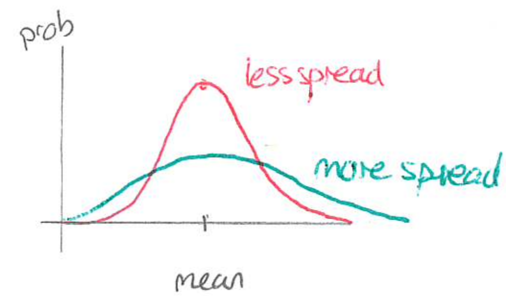
$$r = q^2$$

Then $$\bar{r} = \underbrace{q_1^2 p_1}_{\text{outcome } 1} + \underbrace{q_2^2 p_2}_{2} + \underbrace{q_3^2 p_3}_{3} + \cdots = \sum q_i^2 p_i$$

In general given any function of q, say $f(q)$

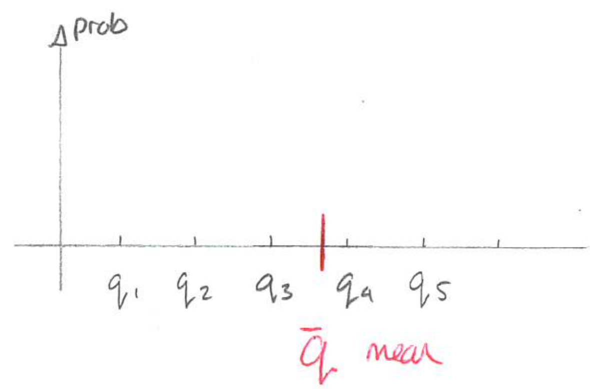
$$\boxed{\overline{f(q)} = \sum_{\text{events } i} f(q_i) p_i}$$

In general probability distributions can assume many forms. However, in many situations (e.g. Galton board, sample averages) the distribution will be peaked in the vicinity of the mean and display some spread around the mean. In the illustrated example the means are the same. However the spread of probabilities around the mean is different. For the distribution with a narrower spread there is a greater degree of confidence in the outcome of any trial. This is quantified via the standard distribution. We form the distance squared between each outcome and the mean:



$$(q_1 - \bar{q})^2, (q_2 - \bar{q})^2, \ldots$$

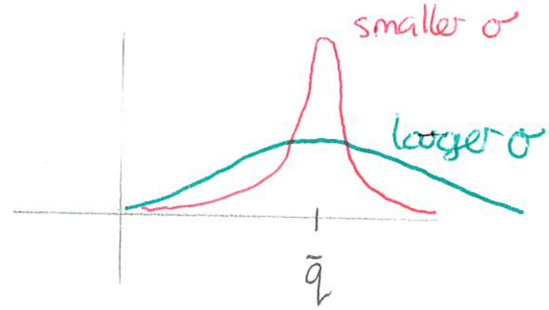
and average these to get the mean squared error (variance)



$$\sum_{\text{events } i} (q_i - \bar{q})^2 p_i$$

Then the standard deviation is:

$$\boxed{\sigma = \sqrt{\sum_{\text{events } i} (q_i - \bar{q})^2 p_i}}$$



A convenient way of calculating this uses

$$\boxed{\sigma = \sqrt{\overline{(q^2)} - (\bar{q})^2}}$$

square all outcomes and average these

find mean and square result

<u>Proof:</u> $\sum_i (q_i - \bar{q})^2 p_i = \sum (q_i^2 - 2q_i\bar{q} + \bar{q}^2) p_i$

$$= \underbrace{\sum q_i^2 p_i}_{\overline{q^2}} - 2\bar{q} \underbrace{\sum p_i q_i}_{\bar{q}} + \bar{q}^2 \underbrace{\sum p_i}_{1}$$

$$= \overline{(q^2)} - (\bar{q})^2 \qquad \blacksquare$$

<u>Example</u> Determine σ for the Galton board problem.

<u>Answer:</u>

i	q_i	p_i	q_i^2
A	12	1/4	144
B	6	1/2	36
C	12	1/4	144

$$\overline{q^2} = \sum p_i q_i^2 = \frac{1}{4}144 + \frac{1}{2}36 + \frac{1}{4}144 = 36 + 18 + 36 = 90$$

$$\bar{q} = \sum p_i q_i = 12/4 + 6/2 + 12/4 = 9$$

$$\sqrt{\overline{q^2} - \bar{q}^2} = \sqrt{90 - 81} = 3$$