Phys 231 2021

Lecture 8

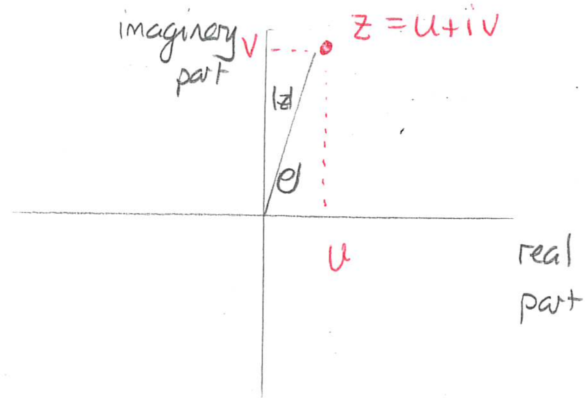
Thurs: Seminar via Zoom (usual ID)

Fri: Read 3.6, 4.1

Mon: HW3

Complex numbers, complex plane

Recall that any complex number can be represented as a single point in the complex plane. We can also represent this number using a distance and angle. The magnitude of the complex number is also known as the modulus of the number.



$$\boxed{\text{If } z = u + iv \text{ then the modulus is } |z| = \sqrt{u^2+v^2}}$$

Along with the modulus we can locate the number in the complex plane by providing the angle θ, measure counterclockwise from the positive real axis. Then:

$$u = |z|\cos\theta$$

$$v = |z|\sin\theta$$

and $$z = |z|(\cos\theta + i\sin\theta)$$

We can then define a complex exponential via:

> For any real θ
>
> $$e^{i\theta} := \cos\theta + i\sin\theta$$

This is the Euler relation / Euler identity. It follows that

> For any complex number
>
> $$z = |z| e^{i\theta}$$
>
> where $|z|$ is the modulus of z
>
> θ is the angle c.c.w. from positive real axis

Quiz 1 → Quiz ~~4~~ 5

We can extract angular trigonometric functions from the complex exponentials using

$$e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

$$\Rightarrow \quad e^{-i\theta} = \cos\theta - i\sin\theta$$

Then adding or subtracting gives

$$\boxed{\begin{aligned} \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \end{aligned}}$$

Separately we can prove that:

$$\boxed{\begin{array}{l}\text{For any real } \theta, \phi \\ 1) \quad e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)} \\ 2) \quad (e^{i\theta})^* = e^{-i\theta} \\ 3) \quad |e^{i\theta}| = 1\end{array}}$$

Proof: Relation 1)

$$e^{i\theta} e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$

$$= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)$$

$$= \cos(\theta+\phi) + i\sin(\theta+\phi)$$

$$= e^{i(\theta+\phi)}$$

Relation 2)

$$(e^{i\theta})^* = (\cos\theta + i\sin\theta)^* = \cos\theta - i\sin\theta = e^{-i\theta}$$

Relation 3)

$$|e^{i\theta}| = |\cos\theta + i\sin\theta| = \underbrace{\sqrt{\cos^2\theta + \sin^2\theta}}_{1} = 1 \quad \blacksquare$$

Complex inverse

We may encounter expressions such as z_1/z_2 : We can reason about division via $z_1/z_2 = z_1 \frac{1}{z_2}$. Then

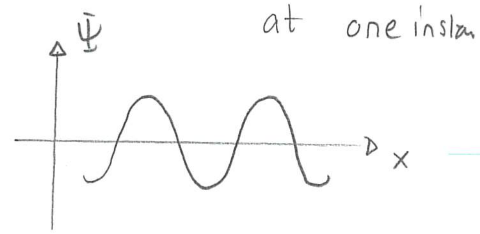
$$z_2 z_2^* = |z_2|^2 \implies z_2 \underbrace{\left(\frac{z_2^*}{|z_2|^2}\right)}_{\text{inverse of } z_2} = 1$$

$$\implies \frac{1}{z_2} = \frac{z_2^*}{|z_2|^2}$$

Complex functions and waves

We represented the simplest type of traveling wave via sinusoidal functions. For example:

$$\Psi(x,t) = A\cos(kx - \omega t - \phi)$$



Interference phenomena can be predicted by adding such waves, which are shifted relative to each other. It turns out that we can obtain the same predictions more easily by representing the wave in terms of complex exponentials. Note that

$$A e^{i(kx-\omega t-\phi)} = \underbrace{A\cos(kx-\omega t-\phi)}_{\text{real part represents actual waves}} + i\,\underbrace{A\sin(kx-\omega t-\phi)}_{\text{complex part, redundant information, helps calculations}}$$

The process will be:

Know wavenumber, k angular frequency, ω phase ϕ, amplitude A → Complex exponential representation $\Psi(x,t) = Ae^{i(kx-\omega t-\phi)}$ → Actual displacement $\mathrm{Re}\left[\Psi(x,t)\right]$

Do calculations here

We will use the complex representation to do actual calculations.

Quiz 6 70%

This shows that for any complex number A, if

$$\Psi(x,t) = A e^{i(kx-\omega t-\phi)}$$

then

$$|\Psi|^2 = |A|^2$$

and if A is real

$$|\Psi|^2 = A^2$$

Applications to light and interference of light

We know that for a light wave described via

$$E = E_0 \cos(kx-\omega t-\phi)$$

the intensity of the light wave is

$$I = \frac{1}{2} c \epsilon_0 E_0^2$$

Now if we used a complex representation

$$E = E_0 e^{i(kx-\omega t-\phi)}$$

then we get

$$I = \underbrace{\frac{1}{2} c \epsilon_0}_{\text{const}} |E|^2$$

complex representation

So

$$\boxed{\begin{array}{l}\text{If } E(x,t) \text{ is the complex representation of a light} \\ \text{wave then the intensity of the light is} \\ \qquad I = \text{constant} \times |E|^2\end{array}}$$

Suppose that we are to add two such waves, usually represented as

$$\left.\begin{aligned} \Psi_1(x,t) &= A\cos(kx-\omega t) \\ \Psi_2(x,t) &= A\cos(kx-\omega t-\phi) \end{aligned}\right\} \Rightarrow \Psi = \Psi_1 + \Psi_2$$

We could use instead

$$\Psi_1(x,t) = Ae^{i(kx-\omega t)}$$

$$\Psi_2(x,t) = Ae^{i(kx-\omega t-\phi)}$$

where A is real

Then

$$\Psi(x,t) = \Psi_1(x,t) + \Psi_2(x,t)$$

$$= Ae^{i(kx-\omega t)} + Ae^{i(kx-\omega t-\phi)}$$

$$= Ae^{i(kx-\omega t)} + Ae^{i(kx-\omega t)}e^{-i\phi}$$

$$= Ae^{i(kx-\omega t)}\left[1+e^{-i\phi}\right]$$

The intensity of this is:

$$I = \text{const} \times |\Psi|^2$$

$$= \text{const} \times \left|Ae^{i(kx-\omega t)}\left[1+e^{-i\phi}\right]\right|^2$$

$$= \text{const} \times \underbrace{|A|^2}_{A^2}\underbrace{|e^{i(kx-\omega t)}|^2}_{1}\underbrace{|1+e^{-i\phi}|^2}$$

$$\rightarrow (1+e^{-i\phi})(1+e^{-i\phi})^*$$

$$= (1+e^{-i\phi})(1+e^{i\phi})$$

$$= 1 + \underbrace{e^{-i\phi} + e^{i\phi}} + \underbrace{e^{-i\phi}e^{i\phi}}_{1}$$

$$= 1 + 2\cos\phi + 1$$

$$= 2(1+\cos\phi)$$

Thus

$$I = \text{const} \times A^2\, 2\,(1+\cos\phi)$$

$$\hookrightarrow \cos\phi = 2\cos^2\frac{\phi}{2} - 1$$

$$= \text{const} \times 4A^2 \cos^2\left[\frac{\phi}{2}\right]$$

we again get that the intensity depends on the shift between the waves.

max intensity $\quad \phi = 0, \pm 2\pi, \pm 4\pi, \ldots$

min " $\quad \phi = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$