Phys 231 2021

Lecture 6

Mon: HW 2 by 5pm

Mon: Read App K complex

Sinusoidal traveling waves

We consider waves the travel in one direction along the x-axis. The most convenient such waves, from a mathematical perspective are sinusoidal waves and these have the form:

$$\boxed{\Psi(x,t) = A\cos(kx - \omega t - \phi)}$$

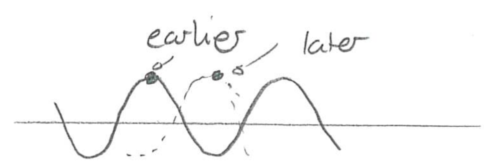
note negative here

where $k = 2\pi/\lambda$

$\omega = 2\pi f$

ϕ = phase (how wave shifts left/right)

By tracking a particular crest we can determine the speed of the wave and a short derivation gives



$$\boxed{v = \omega/k = \lambda f}$$

Quiz 1 90%

Note that a wave which travels with the same speed in the opposite direction could be represented by a similar function with $kx-\omega t-\phi$ replaced by $kx+\omega t-\phi$

So we get:

For sinusoidal waves traveling along the x-axis

$\Psi(x,t) = A\cos(kx - \omega t - \phi)$ travels right $(+x)$

$\Psi(x,t) = A\cos(kx + \omega t - \phi)$ " left $(-x)$

Both travel with speed $v = \omega / k$.

Wave Equation

The sinusoidal traveling wave satisfies the following differential equation:

$$\boxed{\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2}}$$ Wave equation

We can check this: for $\Psi = A\cos(kx - \omega t - \phi)$

$$\frac{\partial \Psi}{\partial x} = -Ak\sin(kx - \omega t - \phi)$$

$$\frac{\partial^2 \Psi}{\partial x^2} = -Ak^2\cos(kx - \omega t - \phi)$$

separately

$$\frac{\partial \Psi}{\partial t} = -(-\omega)A\sin(kx - \omega t - \phi)$$

$$= \omega A\sin(kx - \omega t - \phi)$$

$$\frac{\partial^2 \Psi}{\partial t^2} = -\omega\,\omega A\cos(kx - \omega t - \phi)$$

$$= -\omega^2 A\cos(kx - \omega t - \phi)$$

Substituting into the wave equation gives

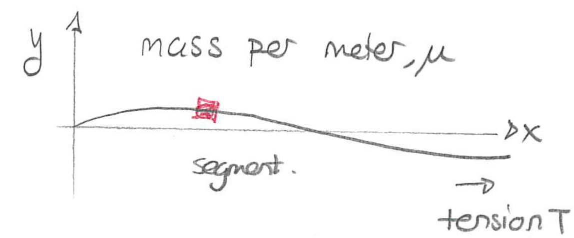
$$\cancel{-}\cancel{A}k^2 \cancel{\cos(kx-\omega t-\phi)} = \frac{1}{v^2}(\cancel{-}\omega^2)\cancel{A}\,\cancel{\cos(kx-\omega t-\phi)}$$

This will be true for all x and t if $\quad k^2 = \dfrac{\omega^2}{v^2}$

$$\Rightarrow \quad v = \omega/k$$

Thus regardless of x and t the function $\Psi(x,t)$ satisfies the wave equation.

This is important because physics often produces something like the wave equation. For example, consider a stretched string. Applying Newton's second law to a segment eventually gives (Openstax section 16.3)



$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

which is the wave equation with $v = \sqrt{T/\mu}$. We then have to solve this for various situations. So we get

Apply physics
e.g. Newton's 2nd Law,
Maxwell's eqns

$\rightarrow$

get wave equation

$$\frac{\partial^2(\text{quantity})}{\partial x^2} = \text{constant} \times \frac{\partial^2(\text{quantity})}{\partial t^2}$$

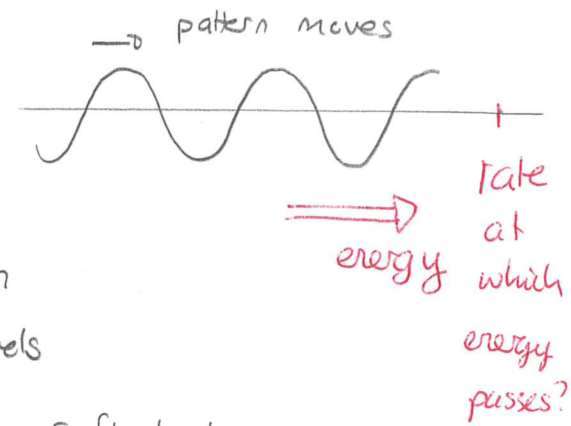
$\swarrow$

Solve via various techniques
Basic solutions:

$$\text{quantity} = A\cos(kx \mp \omega t - \phi)$$

Energy and Intensity

Another aspect of classical waves is that one can verify that in most physical situations the wave transports energy. Typically one can determine an expression for the rate at which energy is transported beyond one point. This is the power delivered by the wave.



The exact expression for this depends on the medium in which the wave travels

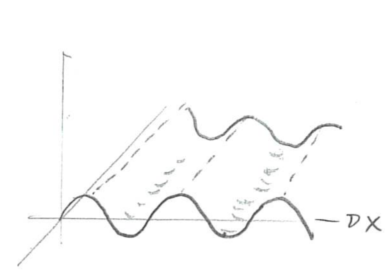
For sinusoidal traveling waves the power fluctuates with time. If we average over a cycle of the wave we always find

> If wave is $\Psi = A\cos(kx - \omega t - \phi)$ then
>
> averaged power = constant $\times A^2$
>
> where the constant depends on the medium.

The simplest types of electromagnetic waves can be described as:

$$\Psi \equiv E(x,t) = E_0 \cos(kx - \omega t - \phi)$$

↳ electric field

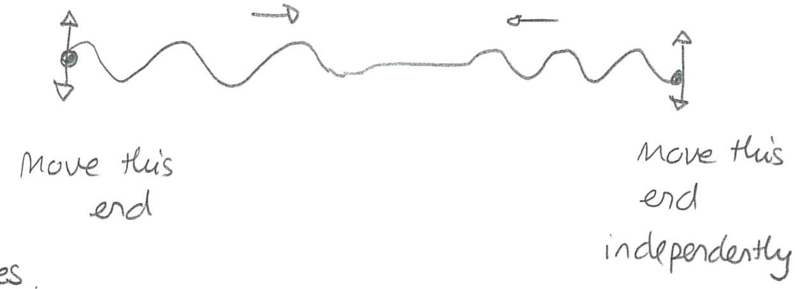


These propagate along x but fill space and so we need to determine the power per unit area. This is the intensity and we get

$$\underbrace{\text{intensity}}_{\text{power/area}} = \underbrace{\tfrac{1}{2}\epsilon_0 c}_{\text{constant}} \underbrace{E_0}_{\text{amplitude}}^2$$

Superposition of Waves

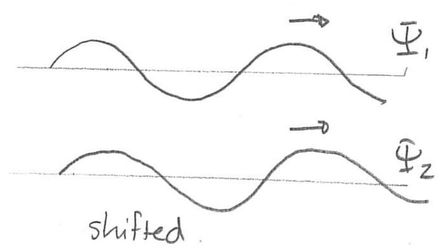
It is possible to simultaneously produce two waves in the same medium. These waves will overlap and produce a single pattern called a superposition of the waves.



Suppose that the waves have the same wavelength (and thus the same frequency). Then suppose they are represented by

$$\Psi_1 = A\cos(kx-\omega t)$$

$$\Psi_2 = A\cos(kx-\omega t-\phi)$$



Then the observed wave will be represented by

$$\Psi(x,t) = \Psi_1(x,t) + \Psi_2(x,t)$$

$$\Psi(x,t) = A\cos(kx-\omega t) + A\cos(kx-\omega t-\phi)$$

This is the superposition of the two waves.

Quiz2 80%

We can add these using a trig identity:

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

Proof: $\cos\alpha = \cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) - \sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

$\cos\beta = \cos\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + \sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

ADD $\Rightarrow$ $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

In this case

$$\Psi(x,t) = 2A\cos\left[\frac{kx-\omega t + kx-\omega t-\phi}{2}\right]\cos\left[\frac{kx-\omega t-(kx-\omega t-\phi)}{2}\right]$$

$$= 2A\cos\left[\frac{2kx-2\omega t-\phi}{2}\right]\cos\left[\frac{\phi}{2}\right]$$

$$= \underbrace{2A\cos\left(\frac{\phi}{2}\right)}_{\text{amplitude}}\underbrace{\cos\left(kx-\omega t-\phi/2\right)}$$

⇒ wave with wavenumber k, frequency ω } same as two waves!

another wave of same type

We see that the amplitude of the combination is

$$A' = 2A\cos\left(\frac{\phi}{2}\right)$$

and this depends on the relative phase

Quiz 3