Modern Physics: Homework 14

Due: 13 May 2021

1 Analysis of the Stern-Gerlach experiment

Suppose that a beam of particles, each with mass M and velocity $\vec{v} = v_x\hat{x}$, enters a region in which the magnetic field is

$$\vec{B} = (600z)\hat{\mathbf{z}}$$

where the units of 600 are Tesla/m. This region extends in the $\hat{x}$ direction for distance L_B. A detector is placed L_D beyond the end of the magnetic field region. The setup is illustrated in Fig. 1. *(231S21)*

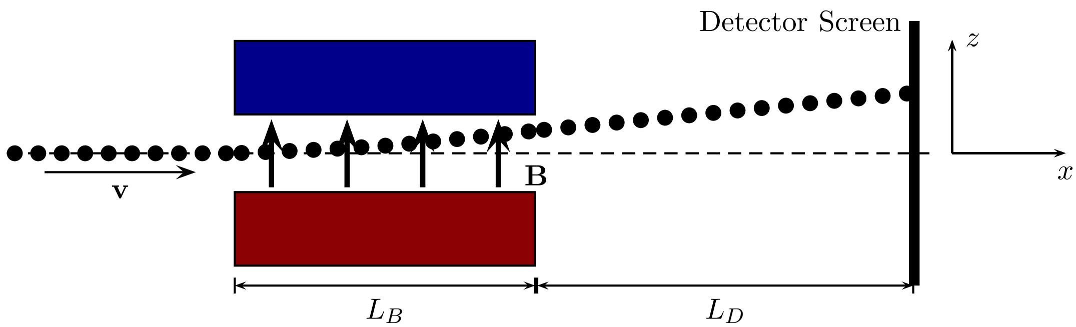

Figure 1: Question 1

a) Suppose that prior to entering the magnetic field, the z component of each particle's magnetic dipole moment has the same value, μ_z. Find an expression, in terms of M and μ_z, for the acceleration of the particles while they are in the region with non-zero magnetic field. Ignore all forces on any particle except that exerted by the magnetic field.

b) Assume that the particles follow trajectories governed by classical mechanics until they hit the screen. Find an expression for the location at which they strike the screen in terms of M, μ_z, v_x, L_B and L_D. Verify that it is proportional to μ_z and inversely proportional to Mv_x^2.

Suppose that the particles have charge q and spin s and are in a state with no orbital angular momentum. The g-factor for the particles is $g = 2$.

c) List the possible values of μ_z in terms of $\hbar, M$ and q and provide expressions for the corresponding deflections in the $\hat{z}$ direction.

d) Determine the distance between "adjacent" deflections, i.e. those corresponding to $\hat{z}$ components of spin adjacent to each other.

e) Suppose each particle produces a spot when it strikes the screen. Suppose that the beam consists of particles whose magnetic dipole moments are randomly oriented. Describe the pattern which will appear on the screen for particles of spin 1/2.

f) Repeat the previous question for particles of spin 1.

g) Repeat the previous question for particles of spin 3/2.

h) Repeat the previous question for particles of spin 2.

i) Suppose that your apparatus is such that $L_B = 0.75\,\text{m}$ and $L_D = 1.25\,\text{m}$. Determine the velocity at which hydrogen atoms, whose mass is $M = 1.67 \times 10^{-27}\,\text{kg}$, must be fired into the magnetic field such that the deflections corresponding to different z components of spin are separated by 2.0 cm. In this case, the magnetic dipole properties of the proton in the hydrogen atom can be ignored; the electron's magnetic dipole is dominant.

2 NMR with protons

The nucleus of a hydrogen atom (a proton) has spin-1/2. The magnitude of the dipole moment of the proton is $1.41 \times 10^{-26}\,\text{J/T}$ (this is the largest value possible for any single component of the dipole moment). In a typical NMR spectrometer, such hydrogen atoms are placed in a magnetic field with magnitude 11.75 T. Determine the energy of the two states available to the proton and the frequency of electromagnetic radiation that is emitted when the proton makes a transition between the two spin states. *(231S21)*

3 Electron spin and orbital motion

An electron is a spin-1/2 particle. Assume that the electron is spherical with a radius no larger than $10^{-18}\,\text{m}$. The possible values of any single component of the spin are $\pm\hbar/2$. Use this to determine the speed of any point of the equator of the electron. This speed must be less than the speed of light. Is it possible that the electron is a solid rotating object? *(231S21)*

4 Two identical particles in an infinite square well

Consider two identical particles, each of mass M, in an infinite square well of length L. The potential is:

$$U(x) = \begin{cases} 0 & 0 \leqslant x \leqslant L \\ \infty & \text{otherwise.} \end{cases}$$

Recall that for a single particle in an infinite square well, the energy eigenfunctions are:

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & 0 \leqslant x \leqslant L \\ 0 & \text{otherwise} \end{cases}$$

where $n = 1, 2, 3, \ldots$. The time-independent Schrödinger equation for the two particles is

$$-\frac{\hbar^2}{2M}\left(\frac{\partial^2\psi(x_1,x_2)}{\partial x_1^2} + \frac{\partial^2\psi(x_1,x_2)}{\partial x_2^2}\right) + U(x_1,x_2)\psi(x_1,x_2) = E\psi(x_1,x_2)$$

where E is the energy of the joint system.

a) Show that $\psi(x_1,x_2) = \psi_m(x_1)\psi_n(x_2)$ satisfies the time-independent Schrödinger equation and determine the corresponding energy eigenvalue, E.

b) Show that

$$\begin{aligned}\psi(x_1,x_2) =& \frac{1}{\sqrt{2}}\left[\psi_m(x_1)\psi_n(x_2) + \psi_n(x_1)\psi_m(x_2)\right] \\ \psi(x_1,x_2) =& \frac{1}{\sqrt{2}}\left[\psi_m(x_1)\psi_n(x_2) - \psi_n(x_1)\psi_m(x_2)\right]\end{aligned}$$

satisfy the time-independent Schrödinger equation for the same energy E as in part (a).

c) Given that the symmetric and antisymmetric wavefunctions in the previous problem have the same energy, one may ask whether there are other measurable quantities whose outcomes provide a distinction between them. Consider position measurements. Show that for both

$$\begin{aligned}\psi(x_1,x_2) =& \frac{1}{\sqrt{2}}\left(\psi_m(x_1)\psi_n(x_2) + \psi_n(x_1)\psi_m(x_2)\right) \\ \psi(x_1,x_2) =& \frac{1}{\sqrt{2}}\left(\psi_m(x_1)\psi_n(x_2) - \psi_n(x_1)\psi_m(x_2)\right)\end{aligned}$$

where $n \neq m$, the expectation values of position measurement outcomes give:

$$\langle x_1 \rangle = \langle x_2 \rangle = \frac{L}{2}.$$

Hints: This may look like a lot of integration but you have done most of the integrals before. Set up the integrals and apply the following results for a single particle in an infinite square well:

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_j^*(x)\psi_k(x)\,dx =& \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases} \\ \int_{-\infty}^{\infty} x|\psi_k(x)|^2\,dx =& \frac{L}{2}.\end{aligned}$$

d) To obtain an observable difference consider the square of the distance between the two particles

$$\left\langle (x_1 - x_2)^2 \right\rangle .$$

Use the results in the hints, which you do not need to prove, to show that this can distinguish between the following states:

$$\begin{aligned}
\psi(x_1, x_2) =& \frac{1}{\sqrt{2}} \left(\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2)\right) \\
\psi(x_1, x_2) =& \frac{1}{\sqrt{2}} \left(\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)\right) .
\end{aligned}$$

Hints: For the states

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}} \left(\psi_1(x_1)\psi_2(x_2) \pm \psi_2(x_1)\psi_1(x_2)\right) ,$$

one can show that

$$\begin{aligned}
\langle x_1 x_2 \rangle =& \frac{L^2}{4} \pm \left(\frac{16L}{9\pi^2}\right)^2 \\
\langle x_1^2 \rangle = \langle x_2^2 \rangle =& \frac{L^2}{3} - \frac{5L^2}{16\pi^2} .
\end{aligned}$$

e) Two particles, either both Bosons or both Fermions are placed in the infinite square well so that one is in the ground state and the other in the first excited state. Describe how you can use the results of position measurement outcomes to decide whether the well contains Bosons or Fermions.

5 Multiple particles in an infinite well

Suppose that six identical, indistinguishable particles, each with mass m, are placed in a one dimensional well with width 2.0×10^{-9} m.

a) Suppose that the particles are electrons and that these do not interact with each other. Determine the lowest possible energy for the collection of particles in the well.

b) Suppose that the particles are neutral kaons (a type of subatomic particle with spin 0) and mass 8.8×10^{-28} kg. Determine the lowest possible energy for the collection of particles in the well.

6 Harris, *Modern Physics, Second Edition*, Ch 8. Prob. 16, page 338.

7 Harris, *Modern Physics, Second Edition*, Ch 8. Prob. 49, page 342.