Modern Physics: Homework 12

Due: 20 April 2021

1 Particle in a two dimensional infinite well

A particle restricted to two dimensions has energy eigenstates that satisfy the two dimensional version of the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right]+U(x,y)\psi(x,y)=E\psi(x,y)$$

The wavefunction can be used to determine probabilities of position measurement outcomes via

$$\text{Prob}[x_0 \leqslant x \leqslant x_1 \text{ and } y_0 \leqslant y \leqslant y_1] = \int_{x_0}^{x_1}\int_{y_0}^{y_1} |\psi(x,y)|^2 \mathrm{d}x\mathrm{d}y.$$

As an example, consider a particle of mass m in the potential

$$U(x,y) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq L \quad \text{and} \quad 0 \leq y \leq L \\ \infty & \text{otherwise.} \end{cases}$$

This is a two dimensional infinite well. *(231S21)*

a) Verify by direct substitution that

$$\psi(x,y) = \begin{cases} A\sin(k_x x)\sin(k_y y) & \text{if} \quad 0 \leqslant x \leqslant L \quad \text{and} \quad 0 \leqslant y \leqslant L \\ 0 & \text{otherwise} \end{cases}$$

where k_x, k_y are constants, is a solution to the time-independent Schrödinger equation.

b) Apply the boundary conditions at $x, y = 0$ and $x, y = L$ to show that these solutions require that

$$k_x = \frac{n_x\pi}{L} \qquad \text{and} \qquad k_x = \frac{n_y\pi}{L}$$

where n_x and n_y are integers. Find an expression for the corresponding energy, E.

c) List all possible states that give the four lowest energies. Are any of these energy levels degenerate?

d) Find the normalization constant, A, by applying

$$\int_0^L\int_0^L |\psi(x,y)|^2 \mathrm{d}x\mathrm{d}y = 1.$$

In the following questions consider the two states: first that for which $n_x = 1$ and $n_y = 2$ and, second, that for which $n_x = 2$ and $n_y = 1$.

e) Is the energy for these two states different or the same?

f) Suppose that a position measurement is performed but that the measuring device has a very poor resolution such that it can only determine whether the particle is in one of eight segments constructed by partitioning the x axis into four equal intervals and the y axis into two equal intervals. Focus on the segment for which $0 \leqslant x \leqslant L/4$ and $0 \leqslant y \leqslant L/2$. Determine the probability with which the particle will be located in this region *for each of the two states.*

g) Does the previous result suggest that the two states are different or not?

2 Particle in a three-dimensional infinite well

Consider a particle in a three dimensional infinite well for which $0 \leqslant x, y, z \leqslant L$. *(231S21)*

a) Consider the following candidates for energies:

$$E = 15\frac{\hbar^2\pi^2}{2mL^2} \quad E = 16\frac{\hbar^2\pi^2}{2mL^2} \quad E = 17\frac{\hbar^2\pi^2}{2mL^2} \quad E = 18\frac{\hbar^2\pi^2}{2mL^2}.$$

Describe which of these are possible energies and, for those which are, which of them are degenerate.

b) Degeneracies usually result from symmetries and for the three-dimensional infinite well these are apparent when the numbers, n_x, n_y, n_z that label the states are rearranged. An accidental degeneracy is one for which a completely different set of numbers gives the same energy. Identify one accidental degeneracy for the three-dimensional infinite well.

3 Particle in a three-dimensional infinite well: probabilities

Consider a particle in a three dimensional infinite well for which $0 \leqslant x, y, z \leqslant L$. Suppose that the particle is in the energy eigenstate

$$\psi(x, y, z) = A \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right)$$

where n_x, n_y, n_z are integers. *(231S21)*

a) Determine the constant A such that the wavefunction is normalized.

b) Determine the probability with which a position measurement will yield an outcome in the region $0 \leqslant x, y, z \leqslant L/2$.

4 Harris, *Modern Physics, Second Edition*, Ch. 7 Prob. 21, page 279.

5 Electron in a well

Suppose that a single electron is trapped in a infinite well, which is known to be either one-dimensional with length L, two-dimensional with sides with length L or three dimensional with edges with length L. In all cases $L = 0.40\,\text{nm}$. By considering the lowest frequency of the radiation emitted in the spectrum, describe how you could determine which type of well was involved. *(231S21)*