# Modern Physics: Homework 7

Due: 16 March 2021

## 1 Uncertainties at the everyday level vs. atomic level

It is always possible to construct wave packets for free particles such that $\Delta x \Delta p = \hbar/2$. This exercise considers the implications of this for two objects: an ordinary sized ball and a proton. In each case the precision with which a particle's position can be known is described by $\Delta x/d$ where $d$ is the size of the object. The smaller this ratio, the more precisely one can know the particle's location. Similarly for the momentum.

First consider a ball of diameter $d_b = 0.02\,\mathrm{m}$, mass $m = 0.01\,\mathrm{kg}$ and with total energy $2\times10^{-9}$ J. Assume that the ball's position is known to a fraction of a typical nuclear diameter, say $\Delta x = 1 \times 10^{-16}\,\mathrm{m}$. Consider the position and momentum of the ball at one instant. *(231S21)*

a) Determine the fractional uncertainty in the ball's position, $\Delta x/d_b$.

b) Determine the momentum of the ball, $p$, and the fractional uncertainty in its momentum, $\Delta p/p$.

c) Can you reasonably say that the ball simultaneously has a position and momentum that are well-defined to within the limits of detectability?

Now suppose that an isolated proton has the same energy and that the uncertainty in its position is the same as for the ball. Extracting the momentum from the energy for this requires the following rule from special relativity: $pc = \sqrt{E^2 - m^2c^4}$.

d) Determine the fractional uncertainty in the proton's position is $\Delta x/d_p$ where $d_p = 1 \times 10^{-15}\,\mathrm{m}$ is the typical nuclear diameter.

e) Use a relativistic rule to determine the proton's momentum and determine the fractional uncertainty in its momentum $\Delta p/p$.

f) Can you reasonably say that the proton simultaneously has a position and momentum that are well-defined to within the limits of detectability?

## 2 Pollen grain

A typical pollen grain from a juniper tree has mass $3.0\times10^{-9}$ kg and diameter $30\,\mu\mathrm{m}$. Suppose that you would like to consider such a pollen grain at one particular location at a moment in time. You would be satisfied to describe its location to an accuracy of 0.01% of its diameter. *(231S21)*

a) If you are able to do this what will be the minimum range of speeds that the grain could have?

b) Is it feasible to describe such a grain of pollen as being at rest at one particular location?

## 3 Particle gun

A "particle gun" fires particles. The intention is to fire particles perfectly horizontally and an example is illustrated. *(231S21)*

a) Which of the following is true, assuming that the gun can be constructed perfectly? Explain your answer.

   i) It is possible that every particle can be fired so that they all travel perfectly horizontally after leaving the gun regardless of the dimensions of the gun.

   ii) It is possible that every particle can be fired so that they all travel perfectly horizontally after leaving the gun but only if the dimensions of the gun are correct.

   iii) It is impossible that every particle can be fired so that they all travel perfectly horizontally after leaving the gun regardless of the dimensions of the gun.

b) Which of the following is true, assuming that the gun can be constructed perfectly? Explain your answer.

   i) It is possible that a single particle can be fired so that it hits a target on the same vertical level as the middle of the gun opening regardless of the dimensions of the gun.

   ii) It is possible that a single particle can be fired so that it hits a target on the same vertical level as the middle of the gun opening but only if the dimensions of the gun are correct.

   iii) It is impossible that a single particle can be fired so that it hits a target on the same vertical level as the middle of the gun opening regardless of the dimensions of the gun.

## 4 Uncertainty principle and energy

The energy of an object connected to a spring is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

where $m$ is the mass of the object and $k$ is the spring constant. *(231S21)*

a) Show that

$$E = \frac{1}{2m}p^2 + \frac{1}{2}kx^2$$

where $p$ is the momentum of the object

b) In classical physics, what is the minimum energy which this system can have?

c) The expectation value of the energy is

$$\langle E \rangle = \frac{1}{2m} \langle p^2 \rangle + \frac{1}{2} k \langle x^2 \rangle .$$

Consider the situation where both the expectation values of the position and momentum are both zero. Show that

$$\langle E \rangle = \frac{1}{2m} (\Delta p)^2 + \frac{1}{2} k (\Delta x)^2.$$

d) Now assuming the minimum uncertainty situation guaranteed by quantum physics determine the minimum energy that the oscillator can have. How does this compare to the classical situation?

## 5 Single slit diffraction and the uncertainty principle

Consider an experiment in which identical particles, each with mass $m$ and momentum $p$ in the vertical direction, are fired vertically toward a barrier containing a single slit of width $w$ as shown in Fig. 1. The possible locations at which any particle can arrive at a collection of detectors placed a distance $L$ from the slit and barrier are denoted $x$ and can be regarded as referring to positions on a horizontal scale fixed with respect to the laboratory.

When the slit is centered beneath the point corresponding to $x = 0$ on the laboratory scale the probability distribution for the location at which the particle arrives, $x$, is

$$P(x) = P_{\text{max}} \frac{\sin^2(\alpha)}{\alpha^2}$$

where, to a good approximation,

$$\alpha = \frac{\pi p w x}{h L}$$

and $P_{\text{max}}$ is a constant.

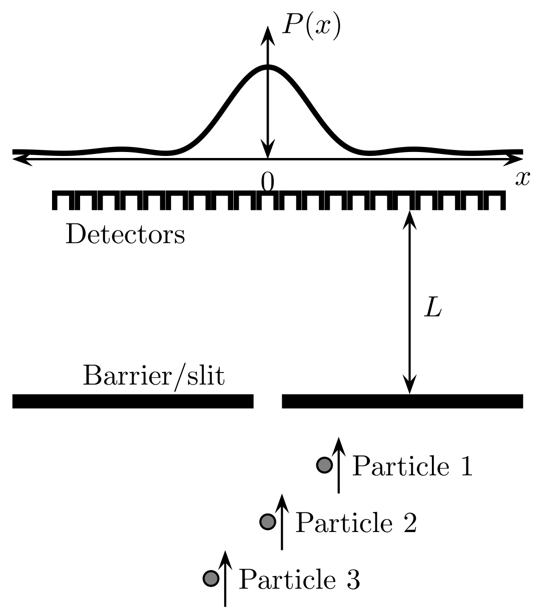


Figure 1: Single slit experiment for particle diffraction.

Recall that the probability that a particle's location will be between $x$ and $x + dx$ is $P(x)dx$, where $P(x)$ is the probability distribution. First consider the position of particles that do pass through the slit. *(231S21)*

a) Suppose that the barrier/slit arrangement can be moved horizontally. If the slit is centered at $x = 0$, and a given particle fired toward the barrier does not arrive on the screen, what can you conclude about its horizontal position at the instant *before* it reached the barrier/slit arrangement? If it did arrive at the screen, what can you conclude about its position at the instant *before* it reached the barrier/slit arrangement?

b) With what uncertainty, $\Delta x$, does the slit determine the horizontal position of the particle at the instant *before* it reached the barrier/slit arrangement? *(Do not invoke Heisenberg's uncertainty principle at this stage.)*

Now consider the horizontal component of the momentum, $p_x$, of particles that pass through the slit.

c) Suppose that the particle which passes through the slit follows a classical trajectory from the instant *after* it passes through the slit until it hits the detectors. Along this trajectory the horizontal, $p_x$, and vertical, $p_y$, components of the particle momentum remain constant. Suppose that the particle arrives at the detector located at $x$. Determine an expression for $p_x$ in terms of $x, L$ and $p_y$ such that this occurs (to simplify the analysis, you can assume that the width of the slit is so small that the particle passed through the point $x = 0$). Assuming the the vertical component of the momentum of the particle is unaffected by the slit, rewrite the expression in terms of the momentum, $p$ prior to arriving at the slit.

d) The range of possible values for $p_x$ can be inferred from the range of arrival locations at the detectors. One measure of the range of arrival locations is the range of all possible locations from $x = 0$ up to the first minimum in the pattern. Show that the maximum value of $p_x$ which results in a particle arriving within this range is $p_x = h/w$.

e) Determine the range of horizontal components of momenta $\Delta p_x$ that describe all arrival locations with the central peak. *(Do not invoke Heisenberg's uncertainty principle at this stage.)* Simplify the expression for $\Delta p_x$ until it only contains $h, \Delta x$ and numbers. Find an expression for $\Delta x \Delta p_x$. Does this agree with Heisenberg's uncertainty principle?

## 6 Free particle wavefunctions

Consider particle with mass $m$ which can move along the $x$ axis but is otherwise free. Determine whether each of the following wavefunctions solves the Schrödinger equation for this particle. *(231S21)*

a) $\Psi(x,t) = A(x + Bt)$ where $A \neq 0$ and $B \neq 0$ are real constants.

b) $\Psi(x,t) = Ae^{-(x-Bt)}$ where $A \neq 0$ and $B \neq 0$ are real constants.

c) $\Psi(x,t) = Ae^{-x^2/4}\, e^{-iEt/\hbar}$ where $A \neq 0$ and $E \neq 0$ are real constants.

d) $\Psi(x,t) = A\sin(kx)\, e^{-iEt/\hbar}$ where $A, k \neq 0$ and $E \neq 0$ are real constants.

## 7 Free particle wavefunctions: non-zero potential

Consider particle with mass $m$ which can move along the $x$ axis. Suppose that the wavefunction that describes the particle is

$$\Psi(x,t) = Ae^{i(px-Et)/\hbar}$$

Determine whether this free particle wavefunction solves the Schrödinger equation for each of the following potentials. *(231S21)*

a) $U(x) = \frac{1}{2}kx^2$ where $k > 0$.

b) $U(x) = kx$ where $k \neq 0$.