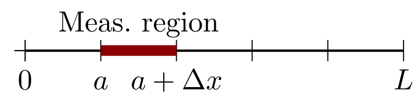
Modern Physics: Homework 6

Due: 9 March 2021

1 Complex exponential wavefunctions

The wavefunction for a particle in a large region $0 \leqslant x \leqslant L$ is

$$\Psi(x,t) = Ae^{i(px/\hbar - \omega t)}$$



within this region and zero outside the region.

Here p is the momentum of the particle. Consider the probability of finding the particle in a "measurement region" of width Δx that is entirely within the range $0 \leqslant x \leqslant L$. Such a measurement region is illustrated and has limits a and $a + \Delta x$. *(231S21)*

a) Which of the following is true? Explain your answer.

   i) The probability of a position measurement yielding an outcome within this measurement region does not depend on the momentum.

   ii) The probability of a position measurement yielding an outcome within this measurement region increases as the momentum increases.

   iii) The probability of a position measurement yielding an outcome within this measurement region decreases as the momentum increases.

b) In each of the following consider various measurement regions, each contained entirely in $0 \leqslant x \leqslant L$. Which of the following is true?

   i) The probability of a position measurement yielding an outcome within this measurement region neither depends on a nor on the region width, Δx.

   ii) The probability of a position measurement yielding an outcome within this measurement region depends on a but not on the region width, Δx.

   iii) The probability of a position measurement yielding an outcome within this measurement region does not depend on a but does depend on the region width, Δx.

   iv) The probability of a position measurement yielding an outcome within this measurement region depends on both a and the region width, Δx.

2 Wavefunction for a confined particle

A particle is confined to the region $-1 \leqslant x \leqslant 1$. At one particular instant the wavefunction that describes the state of the particle is

$$\psi(x) = \begin{cases} A e^{ipx/\hbar}\sqrt{1-x^2} & -1 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

Here $p > 0$. *(231S21)*

a) Determine A such that the wavefunction is normalized.

b) Determine the expectation value for position measurement outcomes.

c) Determine the uncertainty for position measurement outcomes.

d) Determine the probability with which a position measurement gives a positive outcome.

3 Wavefunction for a confined particle

A particle is confined to the region $0 \leqslant x \leqslant L$. At one particular instant the wavefunction that describes the state of the particle is

$$\psi(x) = \begin{cases} Ax & 0 \leqslant x \leqslant L/2 \\ A(-x+L) & L/2 \leqslant x \leqslant L \\ 0 & \text{otherwise} \end{cases}$$

Here $A > 0$. *(231S21)*

a) Sketch the wavefunction and the probability density for position measurement outcomes.

b) Determine A such that the wavefunction is normalized.

c) If the position of the particle is measured is the outcome more likely to be in the region $0 \leqslant x \leqslant L/4$ than it is in the region $L/4 \leqslant x \leqslant L/2$? Explain your answer.

Suppose that you had very many copies of the particle in the same state and measured the position of each.

d) Suppose that the outcomes of the position measurements for the first five particles were all in the range $0 \leqslant x \leqslant L/2$. Does this fact affect the outcome of the position measurement for the sixth particle?

e) Determine the expectation value of the position measurement outcomes.

4 Wavefunctions and position measurements

Consider an ensemble of 10000 identical and independent particles, each restricted to the region $0\,\mathrm{m} \leqslant x \leqslant 1\,\mathrm{m}$. You are guaranteed that at $t = 0$ the particles have been prepared so that they are either *all in the state corresponding to wavefunction*

$$\psi_a(x) = \begin{cases} \sqrt{30}\, x(x-1) & 0\,\mathrm{m} \leqslant x \leqslant 1\,\mathrm{m} \\ 0 & \text{otherwise} \end{cases}$$

or else they are *all in the state corresponding to the wavefunction*

$$\psi_b(x) = \begin{cases} \sqrt{2}\sin(2\pi x) & 0\,\mathrm{m} \leqslant x \leqslant 1\,\mathrm{m} \\ 0 & \text{otherwise} \end{cases}$$

Suppose that you can only perform one position measurement on each particle in the ensemble and are required to determine whether the particles are in the state correspond to wavefunction $\psi_a(x)$ or wavefunction $\psi_b(x)$. This may appear to be possible based on expectation values. *(231S21)*

a) Suppose that the particles are described by $\psi_a(x)$. Plot the probability distribution $P(x)$ for position measurement outcomes. Determine $\langle x \rangle$.

b) Suppose that the particles are described by $\psi_b(x)$. Plot the probability distribution $P(x)$ for position measurement outcomes. Determine $\langle x \rangle$.

c) Is it possible to determine which wavefunction describes the particle based purely on these expectation values of position measurements? Explain your answer.

All position measuring devices have finite resolution. Suppose that initially the measuring device has a (terrible) resolution of $1/2\,\mathrm{m}$. Thus the device can only determine whether each particle is located in one of the following regions:

$$\begin{array}{ll} \text{Region 1:} & 0\,\mathrm{m} \leqslant x \leqslant \frac{1}{2}\,\mathrm{m} \\ \text{Region 2:} & \frac{1}{2}\,\mathrm{m} \leqslant x \leqslant 1\,\mathrm{m}. \end{array}$$

d) For each of the two possible wavefunctions, determine the probability, p_1, that the device produces an output corresponding to region 1 and the probability, p_2, that the device produces an output corresponding to region 2.

e) Given just *one* particle and only the measuring device described above, can you determine which wavefunction describes the particle's state? Can you determine this if you are given all 10000 particles in the ensemble? If so, explain how.

Now suppose that the measuring device is improved and now has a resolution of $1/4\,\mathrm{m}$. Thus it can determine whether each particle is located in one of the following regions:

$$\begin{array}{ll} \text{Region 1:} & 0\,\mathrm{m} \leqslant x \leqslant \frac{1}{4}\,\mathrm{m} \\ \text{Region 2:} & \frac{1}{4}\,\mathrm{m} \leqslant x \leqslant \frac{1}{2}\,\mathrm{m} \\ \text{Region 3:} & \frac{1}{2}\,\mathrm{m} \leqslant x \leqslant \frac{3}{4}\,\mathrm{m} \\ \text{Region 4:} & \frac{3}{4}\,\mathrm{m} \leqslant x \leqslant 1\,\mathrm{m}. \end{array}$$

f) For each of the two possible wavefunctions, determine the probability, p_n, that the device produces an output corresponding to region n for $n = 1, \ldots, 4$. *Hint: It may appear that you will have to calculate four integrals, one for each region. This is not necessary; try to solve the problem with a minimal number of integrations by exploiting various symmetries in the probability density.*

g) Given just *one* particle and only the measuring device described above (i.e. that gives one of four outcomes), can you determine which wavefunction describes the particle's state? Can you determine this with some chance of success if you are given all 10000 particles in the ensemble? Explain your answers.

5 Gaussian wavefunctions

Consider the wavefunction

$$\Psi(x,t) = \frac{1}{\sqrt{a}(2\pi)^{1/4}} e^{-(x-x_0)^2/4a^2}$$

where $a > 0$ and x_0 is a real position. *(231S21)*

a) Verify that this wavefunction is normalized.

b) Show that $\langle x \rangle = x_0$.

c) Show that $\Delta x = a$.

Note: the relevant integrals can be done by looking them up in tables of integrals or using software such as Wolfram Alpha.