Modern Physics: Homework 5

Due: 3 March 2021

1 Probabilities: biased die

Consider an biased die, which returns outcomes $k = 1, \ldots, 6$, each with probabilities p_k where $p_1 = 1/10, p_2 = 1/10, p_3 = 1/10, p_4 = 3/10, p_5 = 3/10, p_6 = 1/10$. *(231S21)*

a) Sketch a graph of the probabilities versus the outcomes of a single die roll.

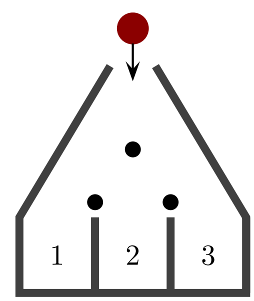
b) Verify that this distribution is normalized (add to one).

c) Determine the mean of the resulting distribution.

d) Determine the standard deviation of the resulting distribution.

2 Plinko probabilities

Consider the Plinko (Galton board) game that was described in class. A two level version is illustrated, with the arrival locations labeled $1, 2, 3$. At each peg the ball can move immediately left or right with equal probability. A three level version is like the two level version but with an additional row of three pegs beneath the lowest row. There will be four outcomes, labeled $1, 2, 3, 4$. *(231S21)*



a) Determine the probabilities of each outcome for the three level version.

b) Determine the mean value of the outcome for the three level version.

c) Determine the standard deviation of the outcome for the three level version.

d) Consider a four level version. List the outcomes and the probabilities with which each occur. Can you spot a pattern in the ratios of the probabilities for the Plinko games with various numbers of levels?

3 Discrete probabilities in physics

An imaginary physical system can be in one of three possible states with energies $0\,\text{J}, 3\,\text{J}, 6\,\text{J}$. The probabilities with which these can occur are $0.75/3, 1/3, 1.25/3$ respectively. *(231S21)*

a) Sketch a graph of the probabilities versus the possible energies.

b) Verify that this distribution is normalized (add to one).

c) Determine the mean of the energy.

d) Determine the standard deviation of the energy.

This is a "toy" model but arrangements of this type occur frequently in statistical physics and can explain many observed bulk physical phenomena.

4 Random number generator

An unbiased random number generator produces numbers $0 \leqslant x \leqslant 1$ such that the probability distribution is

$$P(x) = \begin{cases} A & \text{if} \quad 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

where A is a constant. *(231S21)*

a) Determine A by imposing the condition that the distribution is normalized.

b) Determine the expectation value $\langle x \rangle$ (this is the same as the mean $\overline{x}$.)

c) Determine the standard deviation of the distribution, σ.

d) Calculate the probability that the random generator returns a number within one standard deviation of the mean, i.e. returns a number in the range $\langle x \rangle - \sigma \leqslant x \leqslant \langle x \rangle + \sigma$.

5 Sampling a probability distribution

Consider a random number generator that produces numbers in the range $0 \leqslant x \leqslant 1$. A sample of size N consists of a sequence of numbers produced by the random number generator, $x_1, x_2, \ldots, x_N$. The sample mean is

$$\overline{x}_{\text{samp}} := \frac{1}{N} \sum_{i=1}^{N} x_i$$

and usually constitutes the best way to estimate the expectation value of the distribution, $\langle x \rangle$. However, for a sample of finite size, the sample mean will fluctuate and seldom actually equal the true expectation value. You can investigate this with software such as Excel. *(231S21)*

a) Use your calculator or a computer to generate 5 random numbers and determine the sample mean. Compare this to the expectation value. Repeat this procedure at least twice. Is there substantial fluctuation in the sample averages that you obtain?

b) Use your calculator or a computer to generate 100 random numbers and determine the sample mean. Compare this to the expectation value. Does the sample mean give a better estimate of the expectation value than that generated just using 5 random numbers?

6 Probabilities: continuous outcomes

Suppose that a biased random number generator produces positive real numbers $0 \leqslant x \leqslant \infty$ according to the distribution

$$P(x) = Ae^{-bx}$$

where $b > 0$ is a constant. *(231S21)*

a) Determine an expression for A in terms of b.

b) Determine the expectation value, $\langle x \rangle$, of the resulting distribution.

c) Determine the standard deviation of the resulting distribution.

d) Graph $P(x)$ for $b = 1$ and $b = 5$ and indicated the region within one standard deviation of the expectation value, $\langle x \rangle - \sigma \leqslant x \leqslant \langle x \rangle + \sigma$, in each case. Plot the two graphs using the same vertical and horizontal scales. Compare qualitatively the width and height of the distributions. How are these change as the standard deviation increases?