Modern Physics: Homework 4

Due: 19 February 2021

1 Single slit diffraction

Light, with wavenumber k and angular frequency ω, is incident on a single slit with width a. The light is detected at a point on a distant screen along the direction indicated by the angle θ. The aim of this exercise is to determine the intensity profile of the light on the screen.

The intensity is determined by forming a superposition of waves, one from each point in the slit. Figure 1 illustrates the paths taken by two such waves: one from the lower edge of the slit and another from an arbitrary intermediate point a distance y above the lower edge.

Let r be the distance from the arbitrary intermediate point in the opening to the screen. The wave from this point is

$$\Psi = Ae^{i(kr-\omega t)}.$$

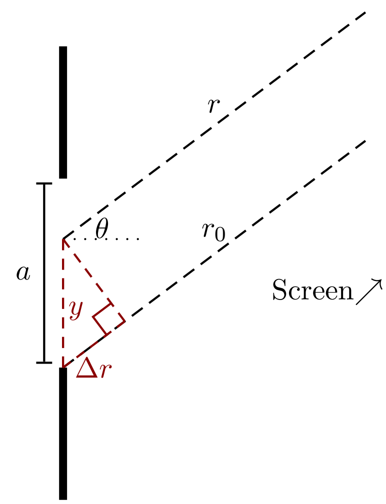

Figure 1: Single slit diffraction.

The superposition will be formed by adding such waves from all points in the opening. This will be done by adding over all possible values of r. This process can be simplified by assuming that the screen is so far away that all the waves travel parallel to each other. The addition will be done with reference to the wave emerging from the lower edge of the slit. *(231S21)*

a) Let r_0 be the distance that the wave from the lower edge travels. The additional distance traveled by the light in the lower path, Δr, is indicated with the triangle outlined in red. Show that $r = r_0 - y\sin\theta$. *(Hint: geometry and trigonometry.)*

b) All such waves emanating from the slit must be added. The result is

$$\Psi = \int_0^a Ae^{i(kr-\omega t)}\mathrm{d}y = Ae^{-i\omega t}e^{ikr_0}\int_0^a e^{-iky\sin\theta}\mathrm{d}y.$$

Evaluate the integral (in terms of k, a and $\sin\theta$) to determine an expression for Ψ.

c) The intensity is $I = |\Psi|^2$. Use this to show that

$$I = A^2a^2\left(\frac{\sin\alpha}{\alpha}\right)^2$$

where

$$\alpha = \frac{\pi a}{\lambda}\sin\theta.$$

d) Suppose that the screen is 0.80 m from the slits and that $a = 5\lambda$. Determine the angle at which the first dark spot appears and the width of the central bright spot in the pattern. Repeat this for $a = 2\lambda$. Did this reduction in slit width make the pattern wider or narrower?

2 Double slit interference: particles

In different experiments particles are fired toward a barrier that contains two slits. Figure 2 illustrates the probabilities with which they arrive at various locations. *(231S21)*

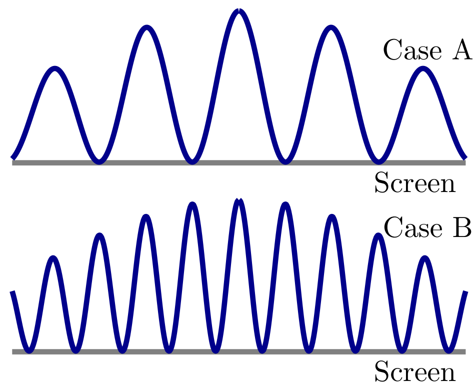

Figure 2: Two experiments.

a) Suppose that, in separate experiments, electrons and protons are fired toward one particular barrier with the same speed. Which of the two diagrams (case A or B) would be the distribution for the electron experiment and which would be the diagram for the proton experiment? Explain your answer.

b) Suppose that in two experiments, neutrons are fired toward the barrier. In one experiment they are all fired at a higher speed, in the other all at a lower speed. Which of the two diagrams (case A or B) would be the distribution for the higher speed experiment and which would be the diagram for the lower speed experiment? Explain your answer.

3 Particle diffraction

Particle diffraction experiments have been done with various particles that are fired toward a grating that effectively consists of many closely spaced slits. Such gratings can be created optically (Phys. Rev. Lett. **87**, 160401 (2001)). A beam of carbon-60 molecules (each contains 60 carbon molecules) is sent toward the grating and is detected on a screen (i.e. particle detectors) behind the grating. The locations at which particles will be most likely to arrive are determined by the same rule as for a double slit. Suppose that the spacing between adjacent slits in the grating is 250 nm. The mass of each molecule is 1.20×10^{-24} kg and the speed with which they approached the grating is 120 m/s. Determine the angles at which the first order locations of high arrival probabilty appear. *The angles are small but can be detected. (231S21)*

4 Single slit diffraction of particles

A beam of particles with momentum p travels in the y direction toward a barrier with a slit of width w as illustrated in Fig. 3. Those particles that pass through the slit travel toward an array of detectors placed at a distance L from the barrier and slits.

The aim of this exercise is to determine the probability with which particles arrice at various locations and compare these predictions to experimental results. *(231S21)*

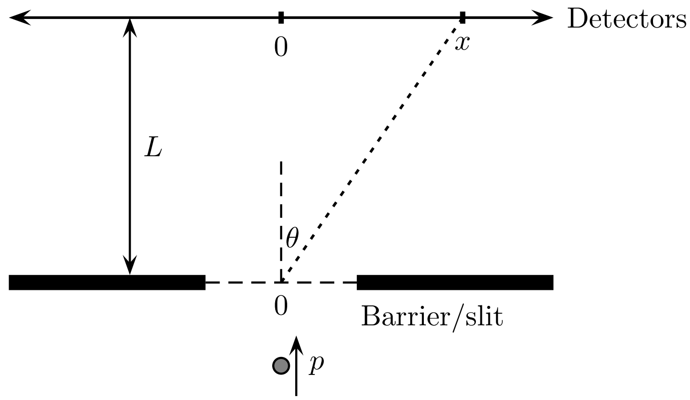

Figure 3: Single slit experiment for particle diffraction.

a) Determine an expression for k in terms of p and $\hbar$.

b) Let w be the width of the slit. Determine the probability, $P(\theta)$, for the detection of particles at the illustrated angle θ (see Fig. 3). *(Hint: Use the known result from the analogous optical situation.)*

c) When the screen is very far from the barrier compared to the slit width $\sin\theta \approx x/L$. Use this to determine an expression for the arrival probability in terms of x as illustrated in Fig. 3.

d) The "side minima" are locations where no particles will arrive. Determine the locations of the first three side minima in terms of L, w, p and $\hbar$.

e) Now compare this to the results for the single slit neutron diffraction experiment describe in the article by Zeilinger et.al., Rev. Mod. Phys., **60**, pages 1067-1073 (1988) (there is a link to this article on the course web page). In this article Fig. 1 shows the experimental setup with the diffracting slit denoted S_5 and the detector S_4. The probability distribution obtained from a single slit is shown in Fig. 2 and Fig. 3 (in that article). In this experiment $w = 90 \times 10^{-6}$ m, $\lambda = 19.26 \times 10^{-10}$ m (see the beginning of section IV) and L can be inferred from Fig. 1. Use these values to determine the location of the first side minimum. How does this compare to the experimental result as shown in Figs. 2 and 3 of the article?

5 Double slit interference of particles

A beam of particles with momentum p travels in the y direction toward a barrier with two slits separated by distance d as illustrated in Fig. 4. Those particles that pass through the slit travel toward an array of detectors placed at a distance L from the barrier and slits. Assume that the distance from the slits to the screen is much larger than the separation between the slits.

Consider the probability density, $P(x)$, for the detection of particles at a distance x from a point immediately above the midpoint between the slits. *(231S21)*

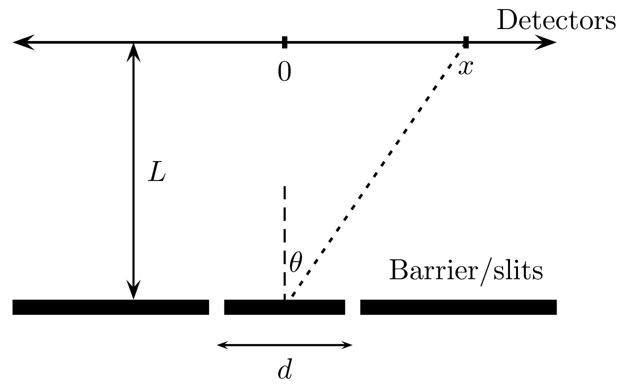

Figure 4: Double slit experiment for particle diffraction.

a) Show that when $x \ll L$,

$$P(x) = P_{\max} \cos^2 \left(\frac{d\pi p x}{hL} \right).$$

Your proof of this result must start from general physical rules that can be used to describe how particle behave when they encounter general barrier and slit arrangements. Provide all the steps needed to reach the final result.

b) Determine an expression for the locations of the first two points beyond $x = 0$ at which particles are most likely to be detected.

c) Now compare this to the results for the double slit neutron diffraction experiment describe in the article by Zeilinger et.al., Rev. Mod. Phys., **60**, pages 1067-1073 (1988) (there is a link to this article on the course web page). In this article Fig. 6 shows the double slit arrangement. The probability distribution obtained from a double slit is shown in Fig. 7. In this experiment $d = 120 \times 10^{-6}$ m, $\lambda = 18.45 \times 10^{-10}$ m (see the beginning of section V) and L can be inferred from Fig. 1. Use these values to determine the location of the first side minimum. How does this compare to the experimental result as shown in Fig. 7?

6 Four level atom

Suppose that a hypothetical atom has a structure similar to the hydrogen atom but with just four electron orbits with energy levels illustrated schematically in Fig 5. The energies are

$$\begin{aligned} E_1 &= -100\,\text{eV} \\ E_2 &= -75\,\text{eV} \\ E_3 &= -50\,\text{eV} \\ E_4 &= -10\,\text{eV} \end{aligned}$$

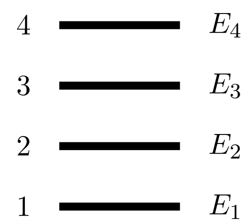

Figure 5: Energy level diagram.

where $1\,\text{eV} = 1.6 \times 10^{-19}$ J. *(231S21)*

a) Indicate all possible ways (transitions between energy levels) in which spectral lines can be produced for this atom (i.e. provide the initial and final energy levels).

b) Calculate the frequencies of all the spectral lines that the atom could produce and indicate the energy transitions which produce each line.

c) Suppose that the electron is in the lowest of these energy levels and is irradiated with light. What frequency of light is necessary to cause the electron to jump into the highest of the energy levels?

d) The atom is irradiated with light of frequency 9.6531×10^{15} Hz. If the electron is initially in the lowest energy level, what will happen to it? If the electron is initially in the third energy level what will happen to it?

e) The atom is irradiated with photons that have energy 100 eV. Explain whether these will be absorbed or not.

f) In separate experiments such atoms are irradiated with photons that either have energy 40 eV or else energy 80 eV. Which photons are more likely to be absorbed? Explain your answer.

g) The atom is irradiated with light of frequency 4×10^{15} Hz. Will this affect the state of the electron? Explain your answer.

7 Bohr model energies

The Bohr model assumes that the electron orbits the nucleus in a circle. Coulomb's law implies that

$$v = \sqrt{\frac{e^2}{4\pi\epsilon_0 m r}}$$

where e is the electron charge, m the electron mass, r the radius of orbit and ϵ_0 the permittivity of free space. The energy of this particle is

$$E = -\frac{e^2}{8\pi\epsilon_0 r}.$$

These are classical results. Bohr's additional assumption was that the angular momentum satisfied

$$L = n\hbar$$

and thus

$$mvr = n\hbar$$

where $n = 1, 2, 3, \ldots$. *(231S21)*

a) Use these to prove that the possible orbital radii are

$$r_n = \frac{4\pi\epsilon_0\hbar^2}{me^2}\, n^2.$$

b) Use these to prove that the possible energies are

$$E_n = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2}\,\frac{1}{n^2}.$$

8 Spectrum predicted by the Bohr model of the hydrogen atom

The Bohr model of the hydrogen atom correctly predicts the energies of the hydrogen atom. *(231S21)*

a) Determine the energies (in units of eV) of the four lowest energy levels of the Bohr atom.

b) Determine the energy (in units of eV) of the highest energy level of the Bohr atom. What is the value of n for this state?

c) How many energy levels are there in the Bohr atom? Explain your answer.

d) Consider transitions in which the energy level drops from *any* state $n > 1$ to the state $n = 1$. Determine the lowest and highest wavelengths possible for all such transitions.

e) Consider transitions in which the energy level drops from *any* state $n > 2$ to the state $n = 2$. Determine the lowest and highest wavelengths possible for all such transitions.

f) The visible spectrum of light has wavelengths in the range 400 nm to 750 nm. Determine the wavelengths of all visible light emitted by the hydrogen atom.