Modern Physics: Homework 3

Due: 15 February 2021

1 Algebraic operations on complex numbers

In the following, let

$$z_1 = 3 + 5i$$
$$z_2 = 4 - 3i$$
$$z_3 = -15 + 10i$$

a) Express $(z_1 + z_2)$ in the form $z = u + iv$ for real u and v.

b) Express $(z_1 - z_2)z_3$ in the form $z = u + iv$ for real u and v.

c) Express $\dfrac{z_1}{z_2}$ in the form $z = u + iv$ for real u and v.

d) Demonstrate, by evaluating each side of the equation separately, that $z_1^* z_2^* = \left[z_1 z_2\right]^*$.

2 The complex plane and the Euler identity

a) Represent the following complex numbers in the complex plane. Use one set of axes for all of them.

$$z_1 = 2$$
$$z_2 = -2i$$
$$z_3 = 1 + 2i$$
$$z_4 = 1 - 2i$$
$$z_5 = -1 + 2i$$
$$z_6 = -1 - 2i$$

b) Consider the product

$$z = \left(2 + 2\sqrt{3}\, i\right)(3 - 3i)$$

Find r and θ such that $z = re^{i\theta}$. *Hint: first represent each of the two complex numbers in the product in terms of magnitude and complex exponential.*

3 Complex modulus

In the following, let

$$z_1 = \sqrt{3} - i$$
$$z_2 = 4 + 3i$$
$$z_3 = 8e^{i3\pi/4}$$

a) Evaluate $|z_1|, |z_2|$ and $|z_3|$.

b) Show by explicitly evaluating each side that $|z_1|^2 = z_1 z_1^*$. Repeat this for z_2.

c) Evaluate $z_1 z_2$ and use the result to show that $|z_1||z_2| = |z_1 z_2|$.

d) Evaluate $z_2 z_3$.

e) Show that it is **not always true** that, for real θ and ϕ,

$$|e^{i\theta} + e^{i\phi}| = |e^{i\theta}| + |e^{i\phi}|.$$

You can show this by finding one example of θ and ϕ so that

$$|e^{i\theta} + e^{i\phi}| \neq |e^{i\theta}| + |e^{i\phi}|.$$

4 Complex exponential functions

For each of the following determine the simplest possible expression for Re$[z]$.

a) $z = 2e^{i(\omega t + \pi/2)}$.

b) $z = Ae^{i\omega t}$ where $A = 3e^{i\pi/4}$.

5 Adding complex exponentials

Let

$$z_1 = 2e^{i(\pi/4+\alpha)}$$
$$z_2 = 2e^{i(\pi/2+\alpha)},$$

where α is real. Let

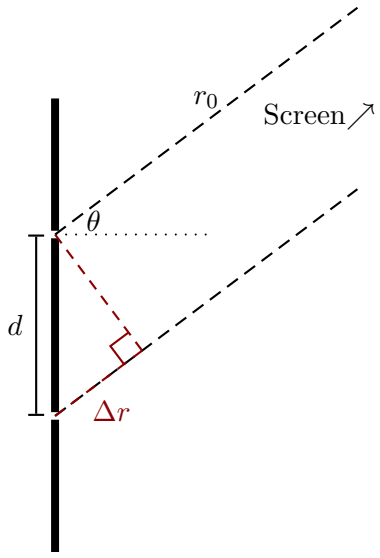
$$z = z_1 + z_2.$$

a) Show that $z = \sqrt{2}e^{i(\alpha+\pi/4)}\left(1 + \sqrt{2} + i\right)$.

b) Evaluate $|z|^2$. Does this depend on α?

6 Double slit interference

Light, with wavenumber k and angular frequency ω, is incident on two slits that are separated by distance d. The light is detected at a point on a distant screen along the direction indicated by the angle θ. Suppose that the screen is very far away compared to the separation between the slits. This means that the paths from each slit to any location on the screen will be very nearly parallel. An example is as illustrated. Let r_0 be the distance from the upper path to the screen and $r = r_0 + \Delta r$ be the distance from the lower path to the screen. The additional distance traveled by the light in the lower path is indicated with the triangle outlined in red. The wave from the upper slit is represented by $\Psi_{\text{upper}} = Ae^{i(kr_0 - \omega t)}$ and that from the lower slit by $\Psi_{\text{lower}} = Ae^{i(kr - \omega t)}$.



a) Use geometry and trigonometry to show that

$$\Delta r = d \sin\theta.$$

b) Determine an expression, using complex exponentials, for the total wave, Ψ that arrives at the screen.

c) Using a rule that the intensity is $I = |\Psi|^2$ show that

$$I = 4A^2 \cos^2\left(\frac{k\Delta r}{2}\right).$$

Note: the identities $\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$ *will be useful.*

d) Show that this gives

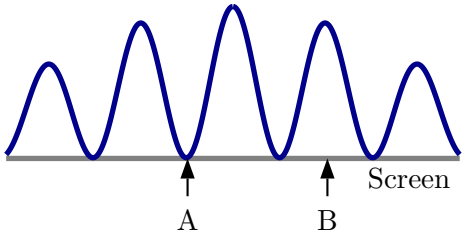
$$I = 4A^2 \cos^2\left(\frac{\pi d \sin(\theta)}{\lambda}\right)$$

e) Use the previous expression to determine the angles at which the intensity will be a maximum.

f) Suppose that $A = 1/2$ and $d = 3\lambda$. Plot the intensity versus angle for all feasible angles θ.

7 Double slit interference: photons

Photons are fired, one at a time toward a barrier that contains two slits. The diagram illustrates that probability with which they arrive at various locations on a screen. The arrows indicate two locations, A and B, on the screen.



a) A photon is fired toward the screen and it hits location B. The next photon is then fired toward the screen. Which of the following is true about the next photon to arrive on the screen? Explain your answer.

i) The photon will definitely not arrive at A and will definitely not arrive at B.

ii) The photon will definitely not arrive at A and will definitely arrive at B.

iii) The photon will definitely not arrive at A and might arrive at B.

iv) The photon will definitely not arrive at B and will definitely arrive at A.

v) The photon will definitely not arrive at B and might arrive at A.

b) Which of the following modifications (select all that could be correct) could cause the next photon to arrive at A? Explain your answer.

i) Increasing the intensity of light.

ii) Decreasing the intensity of light.

iii) Increasing the wavelength of light.

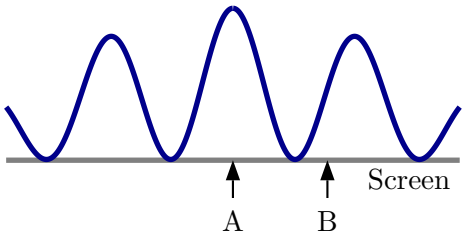
iv) Decreasing the wavelength of light.

v) Increasing the slit separation.

vi) Decreasing the slit separation.

8 Double slit interference: photons

Light with one wavelength is incident on a double slit barrier. The diagram illustrates the intensity of the light at various locations on a screen. The arrows indicate two locations, A and B, on the screen.



a) Consider the statement: "A photon that arrives at A has more energy that a photon that arrives at B." Is this statement true or false? Explain your answer.

b) Suppose that in a short period of time 8000 photons arrive at A. Approximately how many will arrive at B during the same period of time?