Phys 231
Spring 2021

Modern Physics: Class Exam III

30 April 2021

Name: Solution **Total:** **/50**

Instructions

- There are 7 questions on 6 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

$c = 3.0 \times 10^8\,\text{m/s}$ $h = 6.63 \times 10^{-34}\,\text{Js}$ $k_B = 1.38 \times 10^{-23}\,\text{J/K}$ $1\,\text{eV} = 1.6 \times 10^{-19}\,\text{J}$

$m_{\text{electron}} = 9.11 \times 10^{-31}\,\text{kg}$ $m_{\text{proton}} = 1.67 \times 10^{-27}\,\text{kg}$ $m_{\text{neutron}} = 1.67 \times 10^{-27}\,\text{kg}$

Question 1

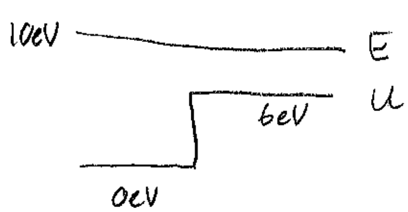
A free particle with energy 10 eV moves to the right. It encounters a step in potential from 0 eV to 6 eV. Which of the following (choose one) is true?

i) If classical physics applies the particle will definitely continue moving right after the step. If quantum physics applies the particle will definitely continue moving right after the step.

ii) If classical physics applies the particle will definitely continue moving right after the step. If quantum physics applies the particle might be reflected and travel left.

iii) If classical physics applies the particle might be reflected and travel left. If quantum physics applies the particle will definitely continue moving right after the step.

iv) If classical physics applies the particle might be reflected and travel left. If quantum physics applies the particle might be reflected and travel left.



classical continues right slows.
quantum could be reflected

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Question 2

~~Protons~~ Electrons with energy 80 eV travel toward a rectangular barrier with height 100 eV and width ~~3.0~~ nm. Determine the fraction of ~~protons~~ electrons that pass through the barrier.

0.50

$$T = 16 \frac{E}{U_0}\left(1-\frac{E}{U_0}\right) e^{-2L\sqrt{2m(U_0-E)/\hbar^2}}$$

$E = e\ 80eV$

$U_0 = e\ 100eV$

$$\sqrt{2m(U_0-E)/\hbar^2} = \sqrt{2 \times 9.11\times10^{-34}\,kg \times 1.6\times10^{-19}\,C \times 20eV / (1.05\times10^{-34}\,J\cdot s)^2}$$

$$= 2.30\times10^{10}\ m^{-1}$$

$$T = 16\ \frac{80}{100}\left(1-\frac{80}{100}\right) e^{-2\times0.5\times10^{-9}\times2.3\times10^{10}}$$

$$= 2.6\times10^{-10}$$

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Question 3

A particle with mass m is in a two-dimensional square well whose sides have length L.

a) Determine the energy and list the states of the lowest degenerate energy level.

$$E = \frac{\hbar^2\pi^2}{2mL^2}(n_x^2+n_y^2)$$

$$E = \frac{5\hbar^2\pi^2}{2mL^2}$$

lowest degenerate $n_x = 1,\ n_y = 2$

$n_x = 2,\ n_y = 1$

+4

b) Describe what quantity or quantities could be measured that would enable one to distinguish between the states of the lowest degenerate energy level. Describe how they would enable one to distinguish between these states.

Could measure position since the wavefunctions will be different +3

$$\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right) \quad \text{vs} \quad \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi y}{L}\right)$$

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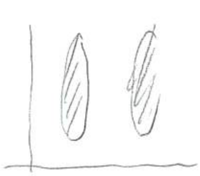
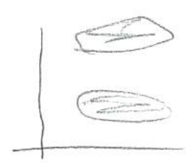


most probable



Question 4

A particle is restricted a three dimensional infinite well for which

$$U(x,y,z) = \begin{cases} 0 & \text{if} \quad 0 \le x \le L \quad \text{and} \quad 0 \le y \le L \quad \text{and} \quad 0 \le z \le L \\ \infty & \text{otherwise.} \end{cases}$$

Consider the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right] + U(x,y,z)\psi(x,y,z) = E\psi(x,y,z)$$

a) Verify by direct substitution that

~~+7~~ +6

$$\psi(x,y,z) = \begin{cases} A\sin(k_x x)\sin(k_y y)\sin(k_z z) & \text{if} \quad 0 \leqslant x,y,z \leqslant L \\ 0 & \text{otherwise} \end{cases}$$

where k_x, k_y and $k_z$are constants, is a solution to the time-independent Schrödinger equation.

~~+3~~

$$\frac{\partial \psi}{\partial x} = +k_x A\cos(k_x x)\sin(k_y y)\sin(k_z z)$$

$$\frac{\partial^2 \psi}{\partial x^2} = -k_x^2 A\sin(k_x x)\sin(k_y y)\sin(k_z z) = -k_x^2\psi$$

Similarly $\frac{\partial^2 \psi}{\partial y^2} = -k_y^2\psi$

$$\frac{\partial^2 \psi}{\partial z^2} = -k_z^2\psi$$

Substitution gives:

$$\frac{-\hbar^2}{2m}\left[-k_x^2 - k_y^2 - k_z^2\right]\cancel{\psi} = E\cancel{\psi} \quad \Rightarrow \quad E = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2)$$

Question 4 continued . . .

b) Determine the possible energies for the particle in terms of $m, L, \hbar, \pi$ and integers.

+8

+6 We need $\Psi(x,y,z) = 0$ when $x = L$ or $y = L$ or $z = L$

$$\Psi(L,y,z) = 0 \Rightarrow A\sin(k_x L)\sin(k_y y)\sin(k_z z) = 0$$

$$\Rightarrow k_x L = n_x \pi \quad \text{for some integer } n_x$$

$$\Rightarrow k_x = \frac{n_x \pi}{L}$$

similarly $k_y = \frac{n_y \pi}{L}$ $\qquad k_z = \frac{n_z \pi}{L}$

$$\Rightarrow E = \frac{\hbar^2 \pi^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

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Question 5

The lowest energy state of the hydrogen atom has radial wavefunction

$$R(r) = \frac{2}{(a_0)^{3/2}}\, e^{-r/a_0}$$

where a_0 is the Bohr radius. This is normalized.

a) Determine the most likely value of r.

$$P(r) = r^2 R^2 = \frac{4}{a_0^3} r^2 e^{-2r/a_0}$$

$$\frac{dP}{dr} = 0 \Rightarrow 2r\, e^{-2r/a_0} + r^2\left(-\frac{2}{a_0}\right) e^{-2r/a_0} = 0$$

$$\Rightarrow 2r - 2r^2/a_0 = 0 \qquad r\,(2 - 2r/a_0) = 0$$

$$\Rightarrow r = a_0$$

b) Determine the expectation value of r.

$$\langle r\rangle = \int_0^\infty r\, P(r)\, dr$$

$$= \int_0^\infty \frac{4}{a_0^3} r^3 e^{-2r/a_0}\, dr = \frac{4}{a_0^3}\frac{3!}{(2/a_0)^4} = \frac{24}{16} a_0$$

$$= \frac{3}{2} a_0$$

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Question 6

A hydrogen atom is in a state for which $l = 3$. List the possible outcomes of a measurement of the z-component of angular momentum.

$L_z = m_\ell \hbar$ $\quad\quad$ $m_\ell = -3, -2, -1, 0, 1, 2, 3.$

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Question 7

Consider a hydrogen atom in a state for which $l \geqslant 1$. Which of the following (choose one) is true?

i) The z-component of the angular momentum could be larger than the magnitude of the angular momentum.
ii) The z-component of the angular momentum could be the same as the magnitude of the angular momentum.
iii) The z-component of the angular momentum will always be less than the magnitude of the angular momentum.

Briefly explain your answer.

$L = \sqrt{\ell(\ell+1)}\,\hbar$

$L_{z\,max} = m_{\ell\,max}\hbar$

$= \ell\hbar < \sqrt{\ell(\ell+1)}\,\hbar$

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