Phys 231
Spring 2021

Modern Physics: Class Exam II

2 April 2021

Name: Solution **Total:** /50

Instructions

- There are 6 questions on 6 pages.
- Show your reasoning and calculations and always explain your answers.

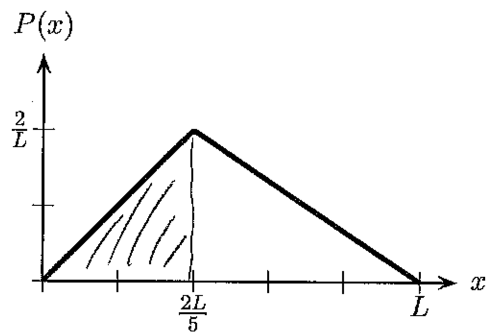
Physical constants and useful formulae

$c = 3.0 \times 10^8\,\mathrm{m/s}$ $h = 6.63 \times 10^{-34}\,\mathrm{Js}$ $k_B = 1.38 \times 10^{-23}\,\mathrm{J/K}$ $1\,\mathrm{eV} = 1.6 \times 10^{-19}\,\mathrm{J}$

$m_{\mathrm{electron}} = 9.11 \times 10^{-31}\,\mathrm{kg}$ $m_{\mathrm{proton}} = 1.67 \times 10^{-27}\,\mathrm{kg}$ $m_{\mathrm{neutron}} = 1.67 \times 10^{-27}\,\mathrm{kg}$

Question 1

The probability density for position measurement outcomes for a particle is as illustrated. Determine the probability with which a position measurement outcome will give $0 \leqslant x \leqslant 2L/5$.



$$\text{Prob} = \int_0^{2L/5} P(x)\,dx$$

$$= \text{area under graph}$$

$$= \frac{1}{2}\left(\frac{2L}{5}\right)\frac{2}{L} = \frac{2}{5}$$

/6

$$P(x) = |\Psi(x)|^2 = \left| \frac{1}{\sqrt{2a}} e^{-i(px/\hbar - \omega t)} \right|^2$$

$$= \frac{1}{2a^2} |e^{\cancel{-ipx/\hbar - \omega t}}|^2$$

$$= \begin{cases} \frac{1}{2a^2} & \text{in } -a < x < a \\ 0 & \text{outside.} \end{cases}$$

Question 2

A particle is known to be in the state described by the wavefunction

$$\Psi(x,t) = \begin{cases} \sqrt{\dfrac{1}{2a}}\, e^{-i(px/\hbar - \omega t)} & -a < x < a \\ 0 & \text{otherwise.} \end{cases}$$

Here p and ω are constants. Which of the following are true about the outcome of a measurement of the particle position done at $t = 0$?

i) It could be any number and all are equally likely.

ii) It could be any number in the range $-a < x < a$ and all are equally likely.

iii) It could be almost any number in the range $-a < x < a$ with the likelihood increasing from left to right.

iv) It could be almost any number in the range $-a < x < a$ with the likelihood decreasing from left to right.

v) It could be almost any number in the range $-a < x < a$ with the likelihood oscillating from left to right.

/4

Question 3

An electron is trapped in an infinite well with width 5.0×10^{-10} m. Determine the two lowest lowest frequencies in the emission spectrum.

$$E_n = n^2 \frac{\hbar^2 \pi^2}{2mL^2} = n^2 \frac{(h/2\pi)^2 \pi^2}{2mL^2} = n^2 \frac{h^2}{8mL^2} = n^2 \frac{(6.63\times10^{-34}\,\text{J·s})^2}{8 \times 9.11\times10^{-31}\,\text{kg}\,(5\times10^{-10}\,\text{m})^2}$$

$$= n^2\, 2.41\times10^{-19}\,\text{J}$$

$$E_{ph} = \Delta E \Rightarrow hf = \Delta E$$

$n=3$

Need smallest transitions: 2→1 and 3→2

$n=2$

For 2→1 $\quad \Delta E = 2^2\, 2.41\times10^{-19}\,\text{J} - 1^2\, 2.41\times10^{-19}\,\text{J}$

$$= 3 \times 2.41\times10^{-19}\,\text{J} = 6.63\times10^{-34}\,\text{J·s}\, f$$

$n=1$

$$\Rightarrow f = 1.09 \times 10^{15}\,\text{Hz}$$

For 3→2 $\quad \Delta E = 3^2(\ldots) - 2^2(\ldots)$

$$= 5(\ldots) = hf$$

/8

$$\Rightarrow f = 1.8 \times 10^{15}\,\text{Hz}$$

Question 4

Consider a particle in an infinite well for which the potential is

$$U(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{otherwise.} \end{cases}$$

The energy eigenstates for the particle are

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & 0 < x < L \\ 0 & \text{otherwise} \end{cases}$$

where n is an integer that labels the state. A quantum lab technician has a collection of identical particles, each its own copy of the same well. The particles are all guaranteed to be in the same energy eigenstate but the value of n for this state is not known. The technician measures the position of each particle and determines both the expectation value and the uncertainty. He saves the expectation value and the uncertainty but then erases all of the individual particle position measurement outcomes and discards the particles. After this he is asked which eigenstate was involved (i.e. what the value of n was).

a) Describe whether the technician can use the expectation value to determine the eigenstate. If so, how would he do this? Explain your answer in terms of calculations or graphs.

We need $\langle x \rangle = \int \Psi^*(x)\, x\, \Psi(x)\, dx$

$$= \int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) x \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L} \int_0^L x \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\left[\frac{x^2}{4} - \frac{x}{4n\pi/L} \sin\left(\frac{2n\pi x}{L}\right) - \frac{1}{8n^2\pi^2/L^2} \cos\left(\frac{2n\pi x}{L}\right)\right]_0^L$$

$$= \frac{2}{L}\left[\frac{L^2}{4} - 0 - \cancel{\frac{1}{8n^2\pi^2/L^2}} + \cancel{\frac{1}{8n^2\pi^2/L^2}}\right] = \frac{L}{2}$$

same for both, cannot tell

Question 4 continued ...

b) Describe whether the technician can use the uncertainty to determine the eigenstate. If so, how would he do this? Explain your answer in terms of calculations or graphs.

Need $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$. Since $\langle x \rangle$ is same regardless of n

we need

$$\langle x^2 \rangle = \int_0^L \psi^*(x)\, x^2 \psi(x)\, dx$$

$$= \frac{2}{L}\int_0^L x^2 \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\left[\frac{x^3}{6} - \frac{x^2}{2n\pi/L}\sin\left(\frac{2n\pi x}{L}\right) - \frac{x}{4n^2\pi^2/L^2}\cos\left(\frac{2n\pi x}{L}\right) + \frac{1}{8n^3\pi^3/L^3}\sin\left(\frac{2n\pi}{L}x\right)\right]_0^L$$

All sin terms evaluate to zero. So

$$\langle x^2 \rangle = \frac{2}{L}\left[\frac{L^3}{6} - \frac{L^3}{4n^2\pi^2}\right] = L^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)$$

This does depend on n, so we could use it.

/12

Question 5

A particle is in a harmonic oscillator potential. The largest wavelength of light that is emitted by the particle is unknown but the second largest is 500 nm. Determine the largest wavelength of light that this oscillator could emit. Describe how the energy levels of the harmonic oscillator explain your answer.

Harmonic oscillator energies.

$n=3$ ——— $7\hbar\omega_0/2$

$n=2$ ——— $5\hbar\omega_0/2$

$n=1$ ——— $3\hbar\omega_0/2$

$n=0$ ——— $\hbar\omega_0/2$

largest

smallest

$$\Delta E_{oscillator} = \frac{hc}{\lambda}$$

$$\Rightarrow \lambda = \frac{hc}{\Delta E_{oscillator}}.$$

Largest $\lambda \Rightarrow$ smallest $\Delta E_{oscillator} = \hbar\omega_0$

Second " $\lambda \Rightarrow$ second smallest " $= 2\hbar\omega_0$

$$\lambda_{largest} = \frac{hc}{\hbar\omega_0}$$

$$\lambda_{\text{second largest}} = \frac{\hbar c}{2\hbar\omega_0} = \frac{1}{2}\lambda_{largest}.$$

So $\lambda_{largest} = 2\lambda_{\text{second largest}} = 1000\,nm$

/8

Question 6

Consider a harmonic oscillator with potential $U(x) = \frac{1}{2} \overset{m\omega_0^2}{\omega_0} x^2$. Consider the following candidate for an energy eigenstate/stationary state

$$\psi(x) = A \sin(kx)$$

where A and k are constants. Determine whether this is a possible energy eigenstate/stationary state for this harmonic oscillator.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega_0^2 x^2 \psi = E\psi.$$

$$\frac{d\psi}{dx} = -kA\cos(kx)$$

$$\frac{d^2\psi}{dx^2} = -k^2 A\sin(kx) = -k^2\psi.$$

Substitute

$$-\frac{\hbar^2}{2m}(-k^2\cancel{\psi}) + \frac{1}{2}m\omega_0^2 x^2 \cancel{\psi} = E\cancel{\psi}$$

$$\frac{k^2\hbar^2}{2m} + \frac{1}{2}m\omega_0^2 x^2 = E$$

$$\Rightarrow \underbrace{\frac{1}{2}m\omega_0^2 x^2}_{\text{depends on } x} = \underbrace{E - \frac{\hbar^2 k^2}{2m}}_{\text{indep of } x}$$

$\Rightarrow$ Not a solution

/12

Modern Physics: Class Integrals

Basic integrals:

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} \qquad \text{if } n \neq -1$$
$$\int \frac{1}{x} \, \mathrm{d}x = \ln x$$
$$\int e^{\alpha x} \, \mathrm{d}x = \frac{e^{\alpha x}}{\alpha}$$

Integrals of Gaussian functions:

$$\int_{-\infty}^{\infty} e^{-(\alpha x^2 + \beta x + \gamma)} \mathrm{d}x = \sqrt{\frac{\pi}{\alpha}} \, e^{\beta^2/4\alpha} \, e^{-\gamma}$$
$$\int_{-\infty}^{\infty} x \, e^{-(\alpha x^2 + \beta x + \gamma)} \mathrm{d}x = \frac{b\sqrt{\pi}}{2\alpha^{3/2}} \, e^{\beta^2/4\alpha} \, e^{-\gamma}$$
$$\int_{-\infty}^{\infty} x^2 \, e^{-(\alpha x^2 + \beta x + \gamma)} \mathrm{d}x = \frac{\sqrt{\pi}}{2\alpha^{3/2}} \left(1 + \frac{\beta^2}{2\alpha}\right) e^{\beta^2/4\alpha} \, e^{-\gamma}$$
$$\int_{-\infty}^{\infty} e^{-\alpha x^2} \mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}$$
$$\int_{-\infty}^{\infty} x \, e^{-\alpha x^2} \mathrm{d}x = 0$$
$$\int_{-\infty}^{\infty} x^2 \, e^{-\alpha x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{2\alpha^{3/2}}$$
$$\int_{0}^{\infty} e^{-\alpha x^2} \mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$
$$\int_{0}^{\infty} x \, e^{-\alpha x^2} \mathrm{d}x = \frac{1}{2\alpha}$$
$$\int_{0}^{\infty} x^2 \, e^{-\alpha x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{4\alpha^{3/2}}$$

provided that $\alpha > 0$. These are valid for any β and γ which could be complex.

Integrals of trigonometric functions:

$$\int \sin(ax)\,\mathrm{d}x = -\frac{\cos(ax)}{a}$$

$$\int \cos(ax)\,\mathrm{d}x = \frac{\sin(ax)}{a}$$

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \qquad \text{if } a \neq b$$

$$\int \sin(ax)\cos(bx)\,\mathrm{d}x = -\frac{\cos((a-b)x)}{2(a-b)} - \frac{\cos((a+b)x)}{2(a+b)} \qquad \text{if } a \neq b$$

$$\int \cos(ax)\cos(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} \qquad \text{if } a \neq b$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \cos^2(ax)\,\mathrm{d}x = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int x\sin(ax)\,\mathrm{d}x = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a}$$

$$\int x\cos(ax)\,\mathrm{d}x = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a}$$

$$\int x\sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\cos((a-b)x)}{2(a-b)^2} + x\frac{\sin((a-b)x)}{2(a-b)} - \frac{\cos((a+b)x)}{2(a+b)^2} - x\frac{\sin((a+b)x)}{2(a+b)} \qquad \text{if } a \neq b$$

$$\int x\sin(ax)\cos(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)^2} - x\frac{\cos((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)^2} - x\frac{\cos((a+b)x)}{2(a+b)} \qquad \text{if } a \neq b$$

$$\int x\cos(ax)\cos(bx)\,\mathrm{d}x = \frac{\cos((a-b)x)}{2(a-b)^2} + x\frac{\sin((a-b)x)}{2(a-b)} + \frac{\cos((a+b)x)}{2(a+b)^2} + x\frac{\sin((a+b)x)}{2(a+b)} \qquad \text{if } a \neq b$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x\cos^2(ax)\,\mathrm{d}x = \frac{x^2}{4} + \frac{x\sin(2ax)}{4a} + \frac{\cos(2ax)}{8a^2}$$

$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

$$\int x^2\cos^2(ax)\,\mathrm{d}x = \frac{x^3}{6} + \frac{x^2}{4a}\sin(2ax) + \frac{x}{4a^2}\cos(2ax) - \frac{1}{8a^3}\sin(2ax)$$