Modern Physics: Class Exam II

17 April 2020

Name: ____________________ **Total:** /40

Instructions

- There are 8 questions on 7 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

$c = 3.0 \times 10^{8}\,\mathrm{m/s}$ $h = 6.63 \times 10^{-34}\,\mathrm{Js}$ $k_B = 1.38 \times 10^{-23}\,\mathrm{J/K}$ $1\,\mathrm{eV} = 1.6 \times 10^{-19}\,\mathrm{J}$

$m_{\mathrm{electron}} = 9.11 \times 10^{-31}\,\mathrm{kg}$ $m_{\mathrm{proton}} = 1.67 \times 10^{-27}\,\mathrm{kg}$

$$\int x^n \, dx = (n+1)x^{n+1}$$

$$\int \sin^2(ax)\, dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int x\sin^2(ax)\, dx = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x\sin(ax)\sin(bx)\, dx = \frac{\cos\left((a-b)x\right)}{2(a-b)^2} + x\frac{\sin\left((a-b)x\right)}{2(a-b)} - \frac{\cos\left((a+b)x\right)}{2(a+b)^2} - x\frac{\sin\left((a+b)x\right)}{2(a+b)} \qquad \text{if} \quad a \neq b$$

$$\int x^2\sin^2(ax)\, dx = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty} xe^{-ax^2}\, dx = 0$$

$$\int_{-\infty}^{\infty} x^2e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}\frac{1}{2a}$$

Question 1

A random number generator is configured so that it produces **one of four numbers.** These numbers are: $2, 3, 4$ and 5. The probabilities with which the random number generator produces each number are all the same. Determine the mean of the numbers produced by this random number generator.

The probabilities are equal at $1/4$ (sum to 1)

$$\bar{x} = \sum p_x x = \frac{1}{4}2 + \frac{1}{4}3 + \frac{1}{4}4 + \frac{1}{4}5$$

$$= \frac{1}{4}(14) = \boxed{3.5}$$

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Question 2

An alpha particle, with mass 6.64×10^{-27} kg is trapped inside a one dimensional infinite well with width 7.0×10^{-15} m. Consider motion along the x axis only.

a) Determine the range of likely momenta for the alpha particle.

$\Delta x \Delta p \geq \hbar/2$ $\qquad \Delta x = 7.0 \times 10^{-15} m$

$$\Delta p = \hbar/2\Delta x = \frac{1.055 \times 10^{-34} J.s}{7.0 \times 10^{-15} m} = 1.5 \times 10^{-20} kgm/s.$$

b) Someone claims that it is impossible that the speed of the alpha particle could be larger than 8000 m/s. Is this claim true or false? Explain your answer?

If this is the case then $\Delta v \leq 8000 m/s$

$$p \simeq m\Delta v = 6.64 \times 10^{-27} kg \times 8000 m/s \leq 8000 m/s$$

$$= 5.3 \times 10^{-23} m/s$$

But Δp is larger so this is not true

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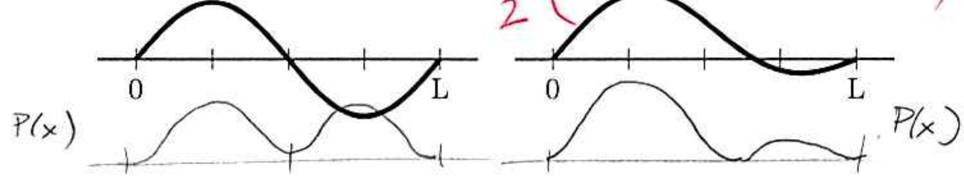
Question 3

Consider a particle in a one dimensional infinite well. The particle is restricted to the range $0 \leqslant x \leqslant L$. At one instant you are told that the wavefunction, $\psi(x)$, could be one of the two displayed below.

$\psi_1 = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)$ $\qquad$ $\psi_2 = \left(\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right) + \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\right)\frac{1}{\sqrt{2}}$



Suppose that you are given one or many particles, each of which could be in one of these two states (and no others). You are guaranteed that **all of the particles are in the same state** but do not know which it is. You have a measuring device which can only determine whether the particle is in the left half of the well ($0 \leqslant x \leqslant L/2$) or the right half of the well ($L/2 \leqslant x \leqslant L$).

a) If you are only given a single particle could you use the measuring device described above to determine in which state it is? Explain your answer. 2

$P(x) = |\psi(x)|^2$ $\qquad$ No. The probability is non-zero for either half. So in each case the particle could be in either half

Prob $= \int_{\text{range}} P(x)\,dx$

b) If you are given one million copies of a single particle (all in the same state) could you use the measuring device described above to determine (with some probability) in which state it is? Explain your answer. 2

YES.

	Prob $(0 \leq x \leq L/2)$	Prob $(\frac{L}{2} \leq x \leq L)$
state ψ_1	$1/2$	$1/2$
ψ_2	$> 1/2$	$< 1/2$

We can tell based on

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Question 4

A quantum harmonic oscillator has potential

$$U(x) = \frac{1}{2} m \omega_0^2 x^2.$$

Consider the following as possible energy eigenstates:

$$\phi(x) = Ax^2$$
$$\psi(x) = Be^{-ax^2}$$

where A, B and a are constants. Check by direct substitution into the time-independent Schrödinger equation whether **each** of these is a possible energy eigenstate/stationary state. If it is, determine the energy associated with the state.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi = E\psi \implies -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2} m\omega_0^2 x^2 \psi = E\psi.$$

2 1) For $\phi(x)$ $\quad \frac{d\phi}{dx} = 2Ax \quad \frac{d^2\phi}{dx^2} = 2A$

Substituting $\implies -\frac{\hbar^2}{2m} 2\cancel{A} + \frac{1}{2} m\omega_0^2 x^2 \cancel{A} x^2 = E \cancel{A} x^2$

will not be true for all x $\implies$ not a solution

5 2) For $\psi(x)$ $\quad \frac{d\psi}{dx} = -2ax\, Be^{-ax^2} \quad \frac{d^2\psi}{dx^2} = -2aB\left[e^{-ax^2} + x(-2ax)e^{-ax^2}\right].$

$$= -2a\left[1 - 2ax^2\right]\psi$$

Substitute $\implies -\frac{\hbar^2}{2m}\left[-2a(1-2ax^2)\right]\cancel{\psi} + \frac{1}{2} m\omega_0^2 x^2 \cancel{\psi} = E\cancel{\psi}$

$$\frac{\hbar^2 a}{2m} + \underbrace{\left[\frac{1}{2} m\omega_0^2 - \frac{2\hbar^2 a^2}{m}\right]}_{=0} x^2 = E \implies E = \frac{\hbar^2 a}{2m}$$

$\implies a^2 = \frac{m^2\omega_0^2}{4\hbar^2}$ $\qquad \implies E = \frac{\hbar\omega_0}{2}$

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$a = \frac{m\omega_0}{2\hbar}$

Question 5

An electron is trapped in a one-dimensional finite well. The largest wavelength emitted by the electron is 750 nm.

2 ~~5~~ a) The largest wavelength emitted corresponds to a transition between two states. Which two states are these?

$|\Delta E| = \frac{hc}{\lambda}$ largest $\lambda \Rightarrow$ smallest ΔE. For well $E_n = \frac{\hbar^2 \pi^2}{2mL^2} n^2$

So smallest ΔE $n=2 \rightarrow n=1$

2 b) Determine the width of the well.

$$\Delta E = E_1 - E_2 = \frac{\hbar^2\pi^2}{2mL^2}(1^2 - 2^2) = -\frac{3\hbar^2\pi^2}{2mL^2} \Rightarrow \frac{hc}{\lambda} = \frac{3\hbar^2\pi^2}{2mL^2}.$$

$$\hbar = \frac{h}{2\pi} \Rightarrow \frac{hc}{\lambda} = \frac{3h^2}{8mL^2} \Rightarrow L^2 = \frac{3h\lambda}{8mc} = \frac{3 \times 6.63\times10^{-34} \times 750\times10^{-9}\,m}{8 \times 9.11\times10^{-31}\,kg \times 3.0\times10^{8}\,m/s}$$

$$L^2 = 6.82\times10^{-19} \Rightarrow L = 8.2\times10^{-10}\,m$$

3 c) Determine the third ~~smallest~~ largest wavelength emitted by the electron.

$$|\Delta E| = \left|\frac{h^2}{8mL^2}(n_i^2 - n_f^2)\right| = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{8mcL^2}{h(n_i^2 - n_f^2)}$$

n_i	n_f	$\lvert n_i^2 - n_f^2 \rvert$	
2	1	3	← largest
3	1	8	
3	2	5	← second
4	1	15	
4	2	12	
4	3	7	← third
5	4	9	

$$\lambda = \frac{8\times9.11\times10^{-31}\,kg \times 3.0\times10^{8}\,m/s \times (8.2\times10^{-10}\,m)^2}{6.63\times10^{-34}\,J\cdot s \cdot 7}$$

$$= 321\,nm$$

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Question 6

A particle is in a one dimensional infinite well with potential

$$U(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{otherwise.} \end{cases}$$

At one particular instant the state of the particle is described by the wavefunction

$$\psi(x) = \begin{cases} \sqrt{\dfrac{30}{L^5}}\, x(x-L) & 0 < x < L \\ 0 & \text{otherwise.} \end{cases}$$

Determine the expectation value, $\langle x \rangle$, for position measurements for this particle.

$$\langle x \rangle = \int_{-\infty}^{\infty} x \, P(x)\, dx \qquad P(x) = |\psi|^2$$

$$= \int_0^L x \, \frac{30}{L^5} \left[x(x-L)\right]^2 dx$$

$$= \frac{30}{L^5} \int_0^L x^3 (x-L)^2 \, dx$$

$$= \frac{30}{L^5} \int_0^L x^3 (x^2 - 2xL + L^2)\, dx$$

$$= \frac{30}{L^5} \int_0^L \left[x^5 - 2x^4 L + x^3 L^2\right] dx$$

$$= \frac{30}{L^5} \left[\frac{x^6}{6} - \frac{2L x^5}{5} + \frac{L^2 x^4}{4}\right]_0^L$$

$$= \frac{30}{L^5} L^6 \left[\frac{1}{6} - \frac{2}{5} + \frac{1}{4}\right] = 30L \, \frac{10 - 24 + 15}{60} = \frac{L}{2}$$

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Question 7

A particle in an infinite well is in its lowest energy state. Determine the expectation value of momentum, $\langle p \rangle$ for this particle.

$$\langle p \rangle = \int_0^L \psi(x)^* \hat{p} \psi \, dx \qquad \psi = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right)$$

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

$$= \int_0^L \frac{2}{L} \sin\left(\frac{\pi x}{L}\right)(-i\hbar)\frac{\partial}{\partial x} \sin\left(\frac{\pi x}{L}\right) dx$$

$$= -\frac{2i\hbar}{L} \underbrace{\int_0^L \sin\left(\frac{\pi x}{L}\right)\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi}{L}x\right) dx}_{=0}$$

$$= 0$$

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Question 8

A proton with kinetic energy 10 eV is incident on a barrier with energy 80 eV. For a particular barrier width, the transmission coefficient for particles is 0.00010. Suppose that the barrier width is tripled (three times what it had been previously). Determine the transmission coefficient for the new barrier width. In both cases the wide barrier approximation is applicable.

$$T = 16 \frac{E_0}{U}\left(1 - \frac{E_0}{U}\right) e^{-2L\sqrt{2m(U_0 - E)}/\hbar}$$

$$= 16 \frac{1}{8}\left(\frac{7}{8}\right) e^{-2L\sqrt{2m(U_0-E)}/\hbar}$$

$$T = \frac{7}{4} e^{-2L\sqrt{2m(U_0-E)}/\hbar} \qquad \text{Let } \alpha = 2\sqrt{2m(U_0-E)}/\hbar$$

$$\Rightarrow T = \frac{7}{4} e^{-L\alpha}$$

Initially $\quad 0.00010 = \frac{7}{4} e^{-L\alpha} \longrightarrow e^{-L\alpha} = \frac{4}{7}(0.00010)$

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Finally $\quad T_f = \frac{7}{4} e^{-3L\alpha}$

$$= \frac{7}{4}\left(e^{-L\alpha}\right)^3 \longrightarrow T_f = \frac{7}{4}\left(\frac{4}{7} 0.00010\right)^3$$

$$= \frac{16}{49}(0.00010)^3 = 3.3 \times 10^{-13}$$