Phys 231
Spring 2020

Modern Physics: Final Exam

13 May 2020

Name: ______________________ **Total:** /~~50~~ 100

Instructions

- There are 14 questions on 11 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

$c = 3.0 \times 10^8\,\mathrm{m/s}$ $\quad h = 6.63 \times 10^{-34}\,\mathrm{Js}$ $\quad k_B = 1.38 \times 10^{-23}\,\mathrm{J/K}$ $\quad 1\,\mathrm{eV} = 1.6 \times 10^{-19}\,\mathrm{J}$

$m_{\mathrm{electron}} = 9.11 \times 10^{-31}\,\mathrm{kg}$ $\quad m_{\mathrm{proton}} = 1.67 \times 10^{-27}\,\mathrm{kg}$ $\quad m_{\mathrm{neutron}} = 1.67 \times 10^{-27}\,\mathrm{kg}$

$$\int x^n\,dx = (n+1)x^{n+1}$$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int x\sin^2(ax)\,dx = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x\sin(ax)\sin(bx)\,dx = \frac{\cos((a-b)x)}{2(a-b)^2} + x\frac{\sin((a-b)x)}{2(a-b)} - \frac{\cos((a+b)x)}{2(a+b)^2} - x\frac{\sin((a+b)x)}{2(a+b)} \qquad \text{if} \quad a \neq b$$

$$\int x^2\sin^2(ax)\,dx = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty} xe^{-ax^2}\,dx = 0$$

$$\int_{-\infty}^{\infty} x^2e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}\frac{1}{2a}$$

In one second $NE_{ph} = P$

$N\frac{hc}{\lambda} = P \Rightarrow N = \frac{P\lambda}{hc}$ product same

$\Rightarrow N$ same

Question 1

Two sources each produce light with a single wavelength. Source A produces light with wavelength 350 nm and power 10 W and source B produces light with wavelength 700 nm and power 5.0 W. Which of the following (choose one) is true regarding the number of photons produced per second by these sources?

i) Source A produces the same number of photons per second as source B.
ii) Source A produces half as many photons per second as source B.
iii) Source A produces one quarter as many photons per second as source B.
iv) Source A produces twice as many photons per second as source B.
v) Source A produces four times as many photons per second as source B.

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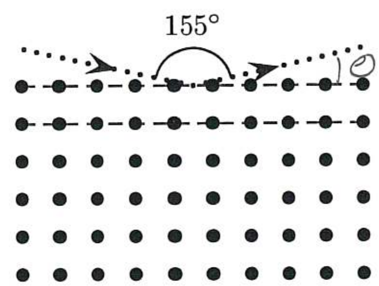
Question 2

In scattering experiments a regular crystal is either bombarded by X-rays or neutrons (each with the same speed). The X-rays or neutrons are scattered off the crystal and the largest angle between the incident and scattered direction is as illustrated. The crystal is cubic with nuclei spaced 3.6×10^{-10} m apart.



$= \frac{180-155}{2}$

$= 12.5°$

a) Determine the wavelength of the X-rays such that scattering occurs as described above.

Bragg scattering: $\underbrace{2d\sin\theta = m\lambda}_{1}$ where m is order $\rightarrow$ smallest $\underbrace{m=1}_{1}$

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$2d\sin\theta = \lambda$

$2 \times 3.6\times10^{-10}\,m \sin 12.5° = \lambda \Rightarrow \lambda = 1.56\times10^{-10}\,m$

b) Determine the speed of the neutrons so that scattering occurs as as described above.

$\underbrace{\lambda = \frac{h}{p}}_{1} = \frac{h}{mv} \Rightarrow \quad v = \frac{h}{m\lambda} = \frac{6.6\times10^{-34}\,J\cdot s}{1.67\times10^{-27}\,kg \times 1.56\times10^{-10}\,m}$

$\underbrace{= 2546\,m/s}_{3}$

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Question 3

Three sources produce waves,each with wavenumber k that combine. It emerges that the resulting superposition at one location and one instant is described by the complex function

$$\psi = Ae^{ikr} + A - Ae^{-ikr}$$

where A and r are real constants.

a) Determine an expression entirely in terms of real quantities and functions for $|\psi|^2$.

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$$\psi = A\left[e^{ikr} - e^{-ikr} + 1\right]$$

$$= A\left[2i\sin kr + 1\right]$$

$$\psi^*\psi = A^2\left[-2i\sin kr + 1\right]\left[2i\sin kr + 1\right]$$

$$= A^2\left[4\sin^2 kr + 1\right]$$

b) Determine *all* possible values for k (in terms of r) such that $|\psi|^2$ is a ~~maximum~~ minimum.

4 Need $\sin^2 kr = \max \Rightarrow kr = n\pi \quad n = \pm 1, \pm 2 \ldots$

$$\Rightarrow k = \frac{n\pi}{r}$$

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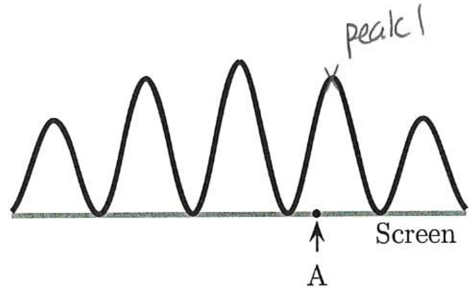
Question 4

Neutrons are fired toward a barrier that contains two slits. For one particular speed and slit spacing the probabilities with which neutrons arrive at a screen are as illustrated. Consider a particular location on the screen labeled A.



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a) Suppose that one particular neutron arrives at location A. The next neutron is fired in exactly the same way toward the barrier/slit arrangement. The person who fires it predicts that this neutron will arrive at A, exactly as its predecessor did. Is this prediction correct or not? Explain your answer. → and nowhere else

No, there are non-zero probabilities of arrival elsewhere.

It could arrive at other locations

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b) A stream of neutrons is fired toward the screen and, while this occurs, their speeds are gradually increased. Does the rate at which neutrons arrive at location A increase, decrease or stay the same when the speed becomes slightly larger than that producing the illustrated pattern? Explain your answer.

Maxima occur where $d\sin\theta = n\lambda$ $n=$ integer

$\lambda = \frac{h}{mv}$ $\Rightarrow$ $d\sin\theta = n\frac{h}{mv}$

$\Rightarrow$ $\sin\theta = n\frac{h}{dmv}$

as v increases θ decreases means peak 1 moves left

$\Rightarrow$ rate increases

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Question 5

A hypothetical artificially created quantum system has four energy states. They are labeled $n = 1, 2, 3, 4$ and their energies are given by $E_n = (10\,\text{eV})n^2$. Determine all possible wavelengths of light (or electromagnetic radiation) that this system can emit.

$$|\Delta E| = E_{ph} = \frac{hc}{\lambda} \qquad \lambda = \frac{hc}{|\Delta E|}$$

Need all ΔE and all λ

$$\Delta E = 10\text{eV}\left(n_f^2 - n_i^2\right) \Rightarrow \lambda = \frac{hc}{\left|10\text{eV}\left(n_f^2 - n_i^2\right)\right|}$$

$$= \frac{6.63\times10^{-34}\,\text{J}\cdot\text{s} \times 3.0\times10^{8}\,\text{m/s}}{10\text{eV}\times1.6\times10^{-19}\,\text{J/eV}\left(n_f^2 - n_i^2\right)}$$

$$= \frac{1.24\times10^{-7}\,\text{m}}{n_f^2 - n_i^2}$$

n_i	n_f	λ
4	3	1.77×10^{-8}m
4	2	1.03×10^{-8}m
4	1	0.83×10^{-8}m
3	2	2.48×10^{-8}m
3	1	1.55×10^{-8}m
2	1	4.14×10^{-8}m

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Question 6

The Bohr model predicts the energies of the hydrogen atom. The ionization energy of the atom is the energy required to have an electron jump from the lowest possible energy state to the highest possible energy state. Determine the ionization energy for the hydrogen atom as predicted by the Bohr model.

$$\left.\begin{aligned} n_i &= 1 \\ n_f &= \infty \\ E &= -13.6\,\text{eV}\ \frac{1}{n^2} \end{aligned}\right\} \Rightarrow \quad \Delta E = -13.6\,\text{eV}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

$$= -13.6\,\text{eV}\left(\frac{1}{\infty^2} - 1\right)$$

$$= 13.6\,\text{eV}$$

/6

Question 7

Two unbiased six sided dice are rolled. Each can give an one outcome from 1, 2, 3, 4, 5 or 6. Determine the probability with which the sum of the outcomes is 7.

All pairs of outcomes equally likely → there are 36 → each has prob 1/36

die A	die B
1	6
2	5
3	4
4	3
5	2
6	1

total 6 → Prob = $\frac{6}{36}$ ⇒ Prob = 1/6

/6

Question 8

A particle is in a one dimensional infinite well with potential

$$U(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{otherwise.} \end{cases}$$

Consider the following candidate for an energy eigenstate:

$$\psi(x) = \begin{cases} Ax(x-L) & 0 < x < L \\ 0 & \text{otherwise} \end{cases}$$

a) Does this candidate satisfy the boundary conditions at $x = 0$ and $x = L$? Explain your answer.

$$\left.\begin{aligned} \psi(0) &= A0(0\,L) = 0 \\ \psi(L) &= AL(L-L) = 0 \end{aligned}\right\} \quad \psi(0) = \psi(L) = 0 \qquad \text{YES.}$$

b) Check whether this candidate satisfies the time-independent Schrödinger equation for a particle in an infinite well.

Inside

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \overset{0}{\cancel{U(x)}}\psi(x) = E\psi(x)$$

$$\frac{d\psi}{dx} = \frac{d}{dx}A(x^2 - Lx) = 2xA - LA$$

$$\frac{d^2\psi}{dx^2} = 2A$$

Substitution:

$$\frac{-\hbar^2}{\cancel{2}m}\cancel{2}\cancel{A} = E\cancel{A}x(x-L)$$

$$-\frac{\hbar^2}{m} = E\underbrace{x(x-L)}$$

contains x will not satisfy for all x.

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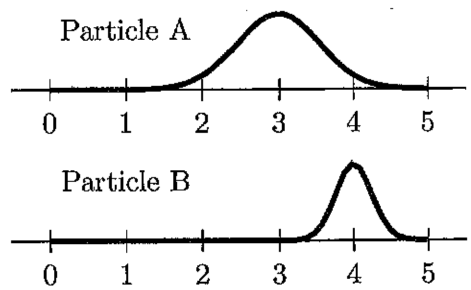
Question 9



Two free particles have the illustrated wavefunctions; the horizontal axes are in units of nm. Suppose that you have many copies of the particles in state A, and many in state B and you measure the momentum of each. Which of the following is true?

i) The range of likely momentum measurement outcomes for A will be about the same as for B.

ii) The range of likely momentum measurement outcomes for A will be larger than for B.

iii) The range of likely momentum measurement outcomes for A will be smaller than for B.

Explain your choice.

$\Delta x \Delta p \geq \hbar/2 \Rightarrow \Delta p \geq \frac{\hbar}{2\Delta x}$

For B $\Delta x_B < \Delta x_A$ for A

Thus $\Delta p_B \geq \Delta p_A$

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Question 10

A particle with mass m is in a three dimensional infinite cubic well whose sides have length L. Determine the number of different states with energy

$$E = 27\frac{\hbar^2\pi^2}{2mL^2}.$$

For a three dimensional infinite well

$$E = \frac{\hbar^2\pi^2}{2mL^2}\underbrace{\left(n_x^2+n_y^2+n_z^2\right)}_{27}$$

Possibilities

n_x	n_y	n_z
5	1	1
1	5	1
1	1	5
3	3	3

4 different states

/5

Question 11

A particle is in a one dimensional infinite well with potential

$$U(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{otherwise.} \end{cases}$$

At one particular instant the state of the particle is described by the wavefunction

$$\psi(x) = \begin{cases} \sqrt{\frac{1}{L}} \sin\left(\frac{\pi x}{L}\right) + \sqrt{\frac{1}{L}} \sin\left(\frac{2\pi x}{L}\right) & 0 < x < L \\ 0 & \text{otherwise.} \end{cases}$$

Determine the probability with which a position measurement will yield an outcome in the range $0 \leqslant x \leqslant L/2$ (i.e. the left half of the well).

$$\text{Prob} = \int_0^{L/2} P(x)\, dx \qquad P(x) = |\psi(x)|^2$$

$$= \frac{1}{L}\left\{ \sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{2\pi x}{L}\right)\right\}^2$$

$$\Rightarrow P(x) = \frac{1}{L}\left\{\sin^2\left(\frac{\pi x}{L}\right) + \sin^2\left(\frac{2\pi x}{L}\right) + 2\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\right\}$$

$$\text{Prob} = \frac{1}{L}\underbrace{\int_0^{L/2} \sin^2\left(\frac{\pi x}{L}\right)dx}_{L/4} + \frac{1}{L}\underbrace{\int_0^{L/2} \sin^2\left(\frac{2\pi x}{L}\right)dx}_{L/4} + 2\frac{1}{L}\int_0^{L/2} \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)dx$$

$$= \frac{1}{L}\frac{L}{2} + \frac{2}{L}\left[\frac{\sin(\pi x/L)}{2\pi/L}\Bigg|_0^{L/2} - \frac{\sin(3\pi x/L)}{2\times 3\pi/L}\Bigg|_0^{L/2}\right\}.$$

$$= \frac{1}{2} + \frac{1}{\pi}\left[1 - \frac{1}{3}(-1)\right]$$

$$= \frac{1}{2} + \frac{4}{\pi 3}$$

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Question 12

The radial part of the hydrogen atom wavefunction for $n = 3, l = 2, m_l = 0$ is

$$R(r) = B\left(\frac{r}{a_0}\right)^2 e^{-r/3a_0}$$

where a_0 is the Bohr radius and B is a normalization constant. Determine an expression for the most likely value of r at which the electron will be found.

Need radial prob density:

$$P(r) = r^2 |R(r)|^2$$

$$= r^2 B^2 \left(\frac{r}{a_0}\right)^4 e^{-r2/3a_0} = \frac{B^2}{a_0^4} r^6 e^{-2r/3a_0}$$

$$\frac{dP}{dr} = 0 \implies \frac{B^2}{a_0^4}\underbrace{\left[6r^5 e^{-2r/3a_0} + r^6\left(-\frac{2}{3a_0}\right)e^{-2r/3a_0}\right]}_{=0} = 0$$

$$\implies r^5\left[6 - \frac{2r}{3a_0}\right]e^{-2r/3a_0} = 0$$

$$\implies 6 = \frac{2r}{3a_0} \implies \boxed{r = 9a_0}$$

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Question 13

A hydrogen atom could be in any one of the states for which $n = 4$.

a) List all possible values of the magnitude of the orbital angular momentum (in terms of $\hbar$).

$L = \hbar\sqrt{l(l+1)}$ $\quad\quad l = 0, 1, \ldots, n-1 = 0, 1, 2, 3$

$$L = \begin{cases} 0\hbar \\ \sqrt{2}\,\hbar \\ \sqrt{6}\,\hbar \\ \sqrt{12}\,\hbar \end{cases}$$

b) List all possible values of the z-component of the orbital angular momentum.

$m_l = -l, \ldots, l$

within this range

$m_l = -3, -2, -1, 0, 1, 2, 3$

$L_z = m_l\hbar$

$= -3\hbar, -2\hbar, -\hbar, 0, \hbar, 2\hbar, 3\hbar$

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Question 14

A spin-5/2 particle is fired into a Stern-Gerlach experiment which deflects it based on the z component of its spin. How many possible deflections could occur for this particle? Explain your answer.

$m_s = -s, -s+1, -s+2, \ldots, s-1, s$

$= -5/2, -3/2, -1/2, 1/2, 3/2, 5/2$

Deflects based on m_s $\rightarrow$ 6 possibilities

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