Fri: HW 15

Tues: Read 10.3.1, 10.3.2

Thurs: Exam II - Ch 8, 9, 10 so far

2012 Q2 Q4

2016 Q1, Q3, Q4

Retarded potentials

Using the Lorentz gauge, the potentials satisfy

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon_0$$

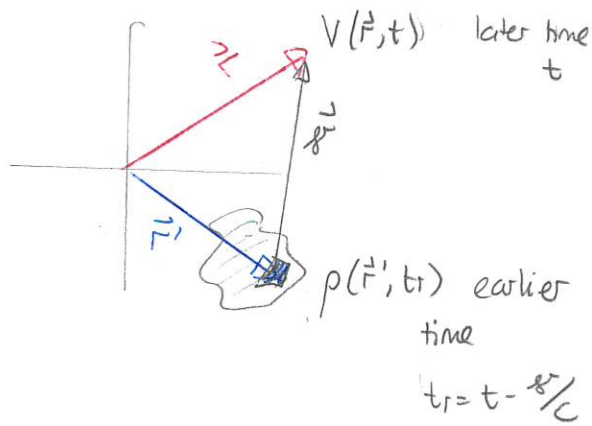
$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

Then these are satisfied by calculating retarded potentials:

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau'$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau'$$

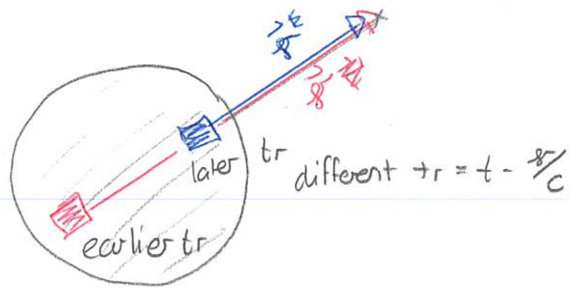
where $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ and the retarded time is $t_r = t - \mathscr{r}/c$



Note that the integration variables, encoded as $\vec{r}\,'$, appear in

1) the spatial argument of the source
2) the denominator $\mathscr{r}$
3) the temporal argument of the source via $\mathscr{r}$ in t_r.

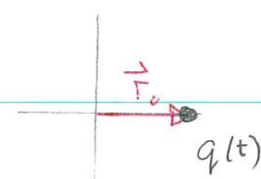
As we range over the entire source domain $\mathscr{r}$ will vary and so will t_r. So different portions contribute at different times. This complicates the integrals.



Point source charges

Consider a point source charge whose location stays fixed but whose magnitude varies with time. If the charge is at $\vec{r}_0$ then the density is:

$$\rho(\vec{r},t) = q(t)\,\delta^3(\vec{r}-\vec{r}_0)$$



where $\delta^3(\vec{r}-\vec{r}_0)$ is the three dimensional Dirac delta function. This has the properties:

1) $\delta^3(\vec{r}-\vec{r}_0) = 0 \quad \text{if } \vec{r} \neq \vec{r}_0$

2) $$\int_{\text{all space}} f(\vec{r})\,\delta^3(\vec{r}-\vec{r}_0)\,d^3r = f(\vec{r}_0)$$

1 Potential from a stationary point charge

Consider a single time varying point charge at the origin. The charge density is

$$\rho(\mathbf{r}', t) = q(t)\delta^3(\mathbf{r}')$$

and the associated current density is

$$\mathbf{J}(\mathbf{r}', t) = -\frac{\dot{q}(t)}{4\pi r'^2}\,\hat{\mathbf{r}}'$$

where $\dot{q}(t)$ is the time derivative of $q(t)$. These are readily shown to satisfy the continuity equation.

a) Determine the retarded potential $V(\mathbf{r}, t)$.

b) Using a symmetry argument, show that the retarded potential $\mathbf{A}(\mathbf{r}, t)$ only has a radial component and that this is independent of angle. Use this result to determine an expression for the magnetic field.

c) Use the magnetic field and one of Maxwell's equations to determine the electric field.

Answer: a)

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,', t_r)}{\mathscr{r}}\,d\tau' \qquad t_r = t - \frac{\mathscr{r}}{c}$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{q(t-\mathscr{r}/c)}{\mathscr{r}}\,\delta^3(\vec{r}\,')\,d\tau'$$

$\leftarrow$ evaluate at $\vec{r}\,'=0$ $\Rightarrow \vec{\mathscr{r}} = \vec{r}$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{q(t-r/c)}{r}$$

Thus the potential at any time depends only on q at the earlier time $t - r/c$

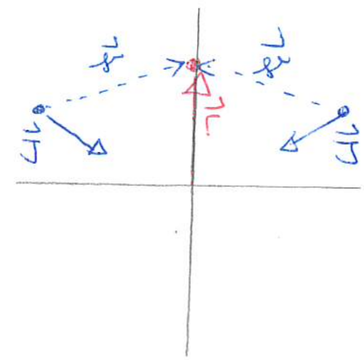
b)

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,', t_r)}{\mathscr{r}}\,d\tau'$$

So we have to sum $\vec{J}$ over all locations.

Suppose that $\vec{r}$ is along the z axis: So

$$\vec{r} = z\hat{z}$$



Then consider the contributions from the two symmetrically located points. Since $\mathscr{r}$ is the same for both the retarded time is the same for both. Then $\vec{J}$ points radially inwards with the same magnitude. Thus only the $\hat{z}$ component remains. This is radially outward.

More precisely

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int -\frac{\dot{q}(t-\mathscr{r}/c)}{4\pi r'^2}\,\frac{\hat{r}'}{\mathscr{r}}\,d\tau'$$

Here
$$\mathscr{r} = \sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}}$$
$$= \sqrt{(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')} = \sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'}$$
$$= \sqrt{z^2 + r'^2 - 2zr'\cos\theta'}$$

$$\Rightarrow \quad \vec{A}(\vec{r},t) = -\frac{\mu_0}{(4\pi)^2}\int_{\text{all space}} \frac{\dot{q}\left(t - \sqrt{z^2+r'^2-2zr'\cos\theta'}/c\right)}{\cancel{r'^2}\left(z^2+r'^2-2zr'\cos\theta'\right)^{1/2}}\,\hat{r}'\,\cancel{r'^2}\sin\theta'\,dr'\,d\phi'\,d\theta'$$

But $\hat{r}' = \cos\phi'\sin\theta'\,\hat{x} + \sin\phi'\sin\theta'\,\hat{y} + \cos\theta'\,\hat{z}$ gives:

$$\vec{A}(\vec{r},t) = -\frac{\mu_0}{(4\pi)^2}\int \frac{\dot{q}(t-\sqrt{\quad}/c)}{\sqrt{\quad}}\left[\cos\phi'\sin\theta'\,\hat{x} + \sin\phi'\sin\theta'\,\hat{y} + \cos\theta'\,\hat{z}\right]\sin\theta'\,dr'\,d\phi'\,d\theta'$$

in $\hat{x}, \hat{y}$

Then the ϕ' terms integrate to zero. Thus:

$$\vec{A}(\vec{r},t) = \frac{-\mu_0}{(4\pi)^2} 2\pi \int_0^{\pi} d\theta' \int_0^{\infty} dr' \frac{\dot{q}(t - \sqrt{\quad}/c)}{\sqrt{r^2 \ldots}} \sin\theta' \cos\theta' \ \hat{z}$$

Converting from $z \to r$ we get

$$\vec{A}(\vec{r},t) = \frac{-\mu_0}{8\pi} \int_0^{\pi} d\theta' \int_0^{\infty} dr' \frac{\dot{q}(t - \sqrt{r^2 + r'^2 - 2rr'\cos\theta'})}{(r^2 + r'^2 - 2rr'\cos\theta')^{1/2}} \sin\theta' \cos\theta' \ \hat{z}$$

The evaluation of the integral depends on the nature of $\dot{q}$ and cannot be done generically. However, we can establish that

$$\vec{A}(\vec{r},t) = A_r(r)\,\hat{r}$$

The magnetic field is:

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta \cancel{A_\phi}) - \cancel{\frac{\partial A_\theta}{\partial\phi}}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\cancel{\frac{\partial A_r}{\partial\phi}} - \frac{\partial}{\partial r}(r\cancel{A_\phi})\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r\cancel{A_\theta}) - \cancel{\frac{\partial A_r}{\partial\theta}}\right]\hat{\phi}$$

$$= 0$$

So $\vec{B} = 0$

c) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{\nabla} \times \vec{E} = 0$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow 0 = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Thus

$$\frac{\partial \vec{E}}{\partial t} = -\frac{1}{\epsilon_0}\vec{J} = \frac{\dot{q}(t)}{4\pi\epsilon_0 r^2}\hat{r}$$

$$\Rightarrow \quad \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q(t)}{r^2}\hat{r}$$

Suppose we had attempted to compute

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$

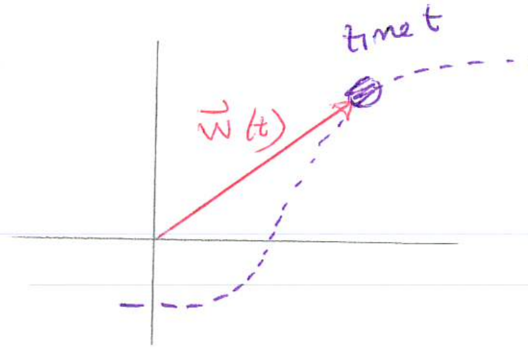
Then $\vec{\nabla}V$ would give a non-zero contribution. So would $\frac{\partial \vec{A}}{\partial t}$.

Both would be along $\hat{r}$. Both contribute.

Potentials from a moving point charge

Now consider a point charge that moves along a known trajectory. Suppose the charge is constant. Again we cannot use Coulomb's Law alone, since there is a non-zero current density. This will result in a magnetic field. We specify the particle trajectory via a time-dependent vector $\vec{w}(t)$.

$$\boxed{\text{The location of the particle at time } t \text{ is } \vec{w}(t)}$$



Then the velocity of the particle is

$$\vec{v}(t) = \dot{\vec{w}}(t)$$

and the acceleration is

$$\vec{a}(t) = \ddot{\vec{w}}(t)$$

The charge density is

$$\boxed{\rho(\vec{r},t) = q\,\delta(\vec{r}-\vec{w}(t))}$$

and the current density is

$$\vec{J} = \rho\vec{v} \Rightarrow \boxed{\vec{J}(\vec{r},t) = q\,\vec{v}(t)\,\delta(\vec{r}-\vec{w}(t))}$$

$\dot{\vec{w}}(t)$

We can now insert these into the retarded potential formalism.

Consider the scalar potential

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}\, d\vec{r}\,'$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{q}{|\vec{r}-\vec{r}\,'|}\,\delta(\vec{r}\,'-\vec{w}(t_r))\, d^3\vec{r}\,'$$

We cannot simply evaluate this by replace $\vec{r}\,'$ in the integrand by $\vec{w}(t_r)$ because $\vec{w}(t_r)$ contains $\vec{r}\,'$ itself. Specifically

$$\delta(\vec{r}\,'-\vec{w}(t_r)) = \delta\left(\vec{r}\,'-\vec{w}\left(t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)\right)$$

and we need to get this in the form $\delta(\vec{r}\,' - \text{something w/o } \vec{r}\,')$ to do the evaluation.

For example if $\vec{w} = \vec{v}_0 t$ then

$$\delta(\vec{r}\,'-\vec{w}(t_r)) = \delta\left(\vec{r}\,'-\vec{v}_0\left(t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)\right)$$

$$= \delta(\underbrace{\vec{r}\,'+\frac{\vec{v}_0}{c}|\vec{r}-\vec{r}\,'|}_{\text{some function of } \vec{r}\,'} - \underbrace{\vec{v}_0 t}_{\text{independent of } \vec{r}\,'})$$

Managing an integral like this requires a co-ordinate transformation

Original co-ordinates:

x', y', z'

$\vec{r}\,' = x'\hat{x} + y'\hat{y} + z'\hat{z}$

$\delta(\vec{r}\,'-\vec{w}(t_r))\, d^3r'$

$\rightsquigarrow$

new co-ordinates

u_x, u_y, u_z

$\vec{u} = u_x\hat{x} + u_y\hat{y} + u_z\hat{z}$

$\delta(\vec{u})\,\frac{1}{J}\, d^3u'$

J depends on u_x, u_y, u_z

The notion would be to try

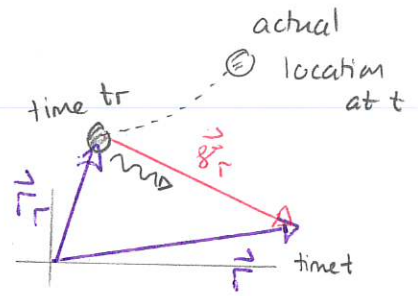
$$\vec{u} = \vec{r}' - \vec{w}(t_r)$$

and this introduces the idea of the <u>retarded position</u>

The retarded position of the particle is the location such that if a signal traveling at the speed of light left that location $\vec{r}'$ at time t_r it would arrive at location $\vec{r}$ at time t.

The retarded position is denoted $\vec{r}_r$ and is defined as

$$\vec{r}_r = \vec{w}(t_r)$$



How could we find the retarded position? We know that the retarded separation vector $\vec{\mathscr{r}}_r$ must at least satisfy

$$\underset{\text{distance signal travels}}{\mathscr{r}_r} = \underbrace{(t - t_r)}_{\text{time to travel}}\ \underset{\text{speed}}{c}$$

Thus

$$|\vec{r} - \vec{r}_r| = c(t - t_r) \Rightarrow |\vec{r} - \vec{w}(t_r)| = c(t - t_r)$$

So we can do:

Find retarded time by solving $|\vec{r} - \vec{w}(t_r)| = c(t - t_r)$ $\rightarrow$ Find retarded position $\vec{r}_r = \vec{w}(t_r)$ $\rightarrow$ Find retarded separation vector $\vec{\mathscr{r}}_r = \vec{r} - \vec{r}_r$