Phys 312 2021

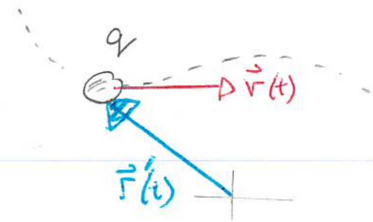
Lecture 17

Fri:

Tues: 10.1.2, 10.1.3

Fields due to moving point charges

A moving point charge is an apparently simple source of electric and magnetic fields. We could imagine a situation where the trajectory of the point charge is well known and specified by $\vec{r}'(t)$. Then the fundamental rules for determining the fields are Maxwell's equations.



$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

The source charge density would be

$$\rho(\vec{r}) = q\,\delta(\vec{r}-\vec{r}'(t))$$

and the source current density

$$\vec{J}(\vec{r}) = \rho(\vec{r})\vec{v}(t) = \rho(\vec{r})\frac{d\vec{r}'}{dt}$$

Both of these are time dependent and therefore the fields would be time-dependent. We would have to solve coupled differential equations. The strategy for solving these would be to convert Maxwell's equations into comparable uncoupled equations involving potentials.

So we will reformulate electromagnetic theory so that

Charge + current densities $\rightarrow$ Determine potentials $V(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ CHOICE!

$\rightarrow$ Fields can be determined by differentiation

We will see that there is considerable choice of potentials which all give rise to the same fields. Then certain allowed choices will uncouple the potentials

charge density $\rho(\vec{r},t)$
current density $\vec{J}(\vec{r},t)$

$\rightarrow$ scalar potential V only determined by ρ SOLVE

$\rightarrow$ vector potential only determined by $\vec{J}$ SOLVE

$\rightarrow$ Determine fields from potentials

This will eventually allow us to determine potentials produced by moving point sources and then use these to determine fields produced by moving point sources.

Potentials in electrostatics + magnetostatics

In electrostatics the potential formulation is as follows:

Time independent charge distribution $\rho(\vec{r}\,')$ → Find electrostatic potential, $V(\vec{r})$, that satisfies Poisson eqn

$$\nabla^2 V(\vec{r}) = -\rho/\epsilon_0$$

Technique

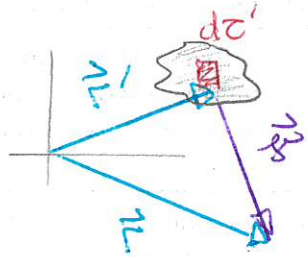
↳ Electric field is

$$\vec{E} = -\vec{\nabla} V(\vec{r})$$

One possibility is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}}\, d\tau'$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$$



In magnetostatics the potential is a vector quantity. The scheme is:

Time independent current distribution $\vec{J}(\vec{r})$ → Magnetic vector potential satisfies

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \quad \text{AND} \quad \vec{\nabla}\cdot\vec{A} = 0$$

Technique

↳ Magnetic field is

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

For localized currents one possibility is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}\, d\tau'$$

This particular scheme results in electric and magnetic fields that can be calculated independently from source charges and currents. In general we expect that this will not be true. To see this suppose that there exist V and $\vec{A}$ such that

$$\vec{E} = -\vec{\nabla} \cdot V \quad \text{and} \quad \vec{B} = \vec{\nabla} \times \vec{A} \quad \text{and} \quad \vec{\nabla} \cdot \vec{A} = 0$$

Substituting into Maxwell's eqns gives:

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \Rightarrow -\vec{\nabla} \cdot \vec{\nabla} V = \rho/\epsilon_0 \Rightarrow \nabla^2 V = -\rho/\epsilon_0 \quad \checkmark$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \underbrace{\vec{\nabla} \times (-\vec{\nabla} V)}_{=0} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \frac{\partial \vec{B}}{\partial t} = 0$$

Not always true.

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0 \Rightarrow \vec{\nabla} \cdot \vec{B} = 0$$

always true

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{J} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

only true if $\frac{\partial \vec{E}}{\partial t} = 0$.

Thus the formalism requires time independent electric and magnetic fields. We need to modify this for general fields

General potential formalism

In all cases a constraint on the fields is

$$\vec{\nabla}\cdot\vec{B} = 0$$

This will always be satisfied if

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

So we assume that there exists such a vector potential $\vec{A}$. The other constraint is:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$= -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A}$$

$$\Rightarrow \quad \vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0$$

This suggests that:

For a given $\vec{A}$ there exists a scalar potential V such that

$$\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}V$$

This would guarantee that $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$. So we have reached

There exists a scalar potential V and a vector potential $\vec{A}$ so that

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

Such potentials guarantee, via vector calculus, that

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

We can also see that such an arrangement will couple the electric and magnetic fields whenever $\frac{\partial \vec{A}}{\partial t} \neq 0$. This

Whenever the vector potential is time dependent then the electric and magnetic fields will be coupled.

The remaining task is to determine equations that relate the scalar and vector potentials to the source charges and currents.

1 General potentials in electromagnetism

The general potentials for electromagnetism allow for calculation of the fields via

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}.$$

a) Substitute into

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

to obtain a differential equation for the potentials in terms of the source charge density.

b) Substitute into

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

to obtain another differential equation for the potentials in terms of the source current density.

Answer: a) $\vec{\nabla}\cdot\vec{E} = -\vec{\nabla}\cdot\vec{\nabla}V - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = \rho/\epsilon_0$

$$= -\nabla^2 V - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}$$

$$\Rightarrow \quad \nabla^2 V + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\rho/\epsilon_0$$

b) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left[-\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}\right]$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} + \mu_0\epsilon_0\vec{\nabla}\left(\frac{\partial V}{\partial t}\right) - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\Rightarrow \nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left[\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right] = -\mu_0\vec{J}$$

Thus we reach:

The scalar potential, V, and vector potential, $\vec{A}$, satisfy

$$\nabla^2 V + \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A}\right) = -\rho/\epsilon_0$$

$$\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \vec{J}$$

Then the scheme is:

Given charge density $\rho(\vec{r},t)$ and current density $\vec{J}(\vec{r},t)$

$\downarrow$

Find potentials $V, \vec{A}$ that satisfy

$$\nabla^2 V + \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A}\right) = -\rho/\epsilon_0$$

$$\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \vec{J}$$

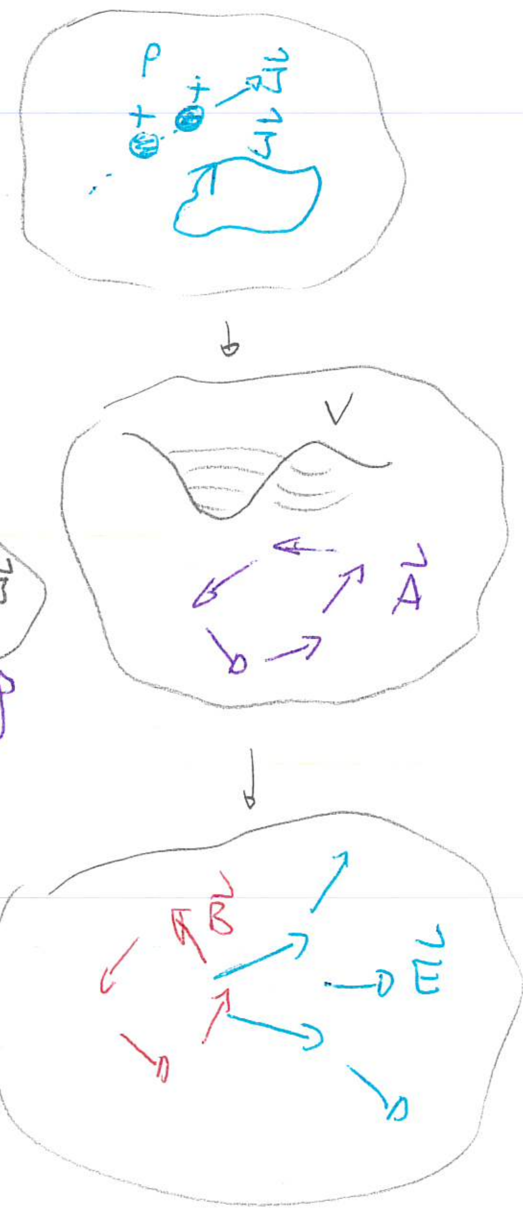
What choices?

$\downarrow$

Determine fields via

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$



2 Fields from potentials

Suppose that in cylindrical coordinates,

$$V = 0$$

$$\mathbf{A} = \frac{\alpha}{s} e^{-(z-vt)^2/a^2}\hat{\mathbf{z}}$$

where α is a constant, $v < c$ is a constant and the vector potential is only non-zero for $s > s_0$.

a) Determine the electric and magnetic fields associated with these potentials

b) Determine the charge and current densities associated with these potentials.

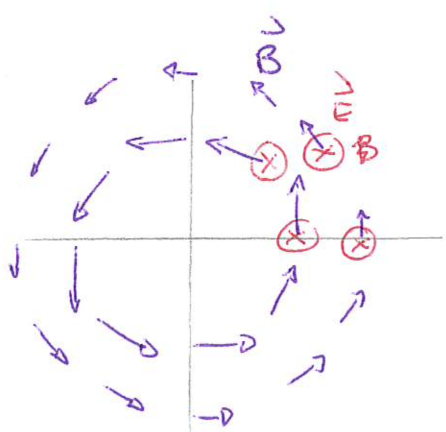
Answer: a) $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$

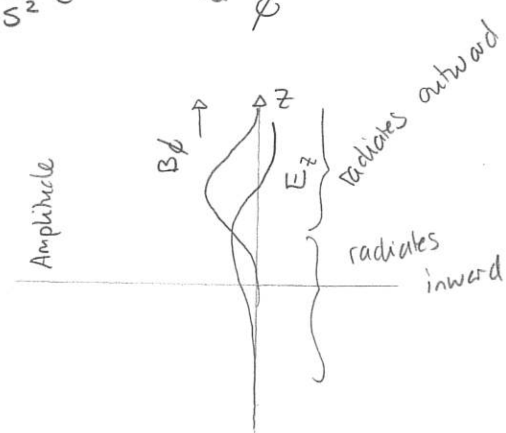
$$= -\frac{\alpha}{s} \frac{\partial}{\partial t}\left[e^{-(z-vt)^2/a^2}\right] \hat{z} \qquad \text{if } s > s_0$$

$$= -\frac{\alpha}{s} \frac{(-v)}{a^2}\left(-2(z-vt)\right) e^{-(z-vt)^2/a^2} \hat{z}$$

$$= -\frac{2\alpha v}{a^2 s} (z-vt)\, e^{-(z-vt)^2/a^2} \hat{z}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = -\frac{\partial A_z}{\partial s}\hat{\phi} = \frac{\alpha}{s^2} e^{-(z-vt)^2/a^2} \hat{\phi}$$





b) $\nabla^2 V = 0$

$$\vec{\nabla}\cdot\vec{A} = \cancel{\frac{1}{s}\frac{\partial}{\partial s}(sA_s)} + \cancel{\frac{1}{s}\frac{\partial A_\phi}{\partial\phi}} + \frac{\partial A_z}{\partial z} \quad \Bigg\} \quad \rho = -\epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})$$

$$= \frac{\alpha}{s}\,\frac{-2(z-vt)}{a^2}\,e^{-(z-vt)^2/a^2} = -\frac{2\alpha}{sa^2}(z-vt)e^{-(z-vt)^2/a^2}$$

$$\frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial t} = -\frac{2\alpha}{sa^2}\frac{\partial}{\partial t}\left[(z-vt)e^{-(z-vt)^2/a^2}\right] = \frac{-2\alpha}{sa^2}\left[-v + \frac{2(z-vt)^2}{a^2}v\right]e^{-(z-vt)^2/a^2}$$

$$\Rightarrow \quad \rho = -\frac{2\alpha\epsilon_0}{sa^2}v\left[1 - \frac{2(z-vt)^2}{a^2}\right]e^{-(z-vt)^2/a^2}$$

↑ propagates

Then

$$\vec{J} = -\frac{1}{\mu_0}\left[\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right)\right]$$

Now

$$\nabla^2\vec{A} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\vec{A}}{\partial s}\right) + \cancel{\frac{1}{s^2}\frac{\partial^2\vec{A}}{\partial\phi^2}} + \frac{\partial^2\vec{A}}{\partial z^2}$$

$$= \frac{1}{s}\frac{\partial}{\partial s}\left(s\left(-\frac{\alpha}{s^2}\right)e^{-(z-vt)^2/a^2}\hat{z}\right) + \frac{\alpha}{s}\frac{\partial^2}{\partial z^2}e^{-(z-vt)^2/a^2}\,\hat{z}$$

$$= \frac{\alpha}{s^3}e^{-(z-vt)^2/a^2}\hat{z} + \frac{\alpha}{s}\frac{\partial}{\partial z}\left(-\frac{2(z-vt)}{a^2}e^{-(z-vt)^2/a^2}\right)\hat{z}$$

$$= \frac{\alpha}{s^3}e^{-(z-vt)^2/a^2}\hat{z} - \frac{2\alpha}{a^2 s}\left[e^{-(z-vt)^2/a^2}\left(1 - \frac{2}{a^2}(z-vt)^2\right)\right]\hat{z}$$

$$= \frac{\alpha}{s}e^{-(z-vt)^2/a^2}\left[\frac{1}{s^2} - \frac{2}{a^2}\left(1 - \frac{2(z-vt)^2}{a^2}\right)\right]\hat{z}$$

Then

$$\frac{\partial\vec{A}}{\partial t} = +\frac{2\alpha v}{sa^2}(z-vt)e^{-(z-vt)^2/a^2}\hat{z}$$

$$\frac{\partial^2\vec{A}}{\partial t^2} = \frac{2\alpha v}{sa^2}e^{-(z-vt)^2/a^2}\left[-v + \frac{2}{a^2}(z-vt)^2 v\right]\hat{z} = -\frac{2\alpha v^2}{sa^2}\left(1 - \frac{2(z-vt)^2}{a^2}\right)\hat{z}$$

Now

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) = \vec{\nabla}\left(-\frac{2\alpha}{sa^2}(z-vt)\,e^{-(z-vt)^2/a^2}\right)$$

$$= \frac{2\alpha}{s^2a^2}(z-vt)\,e^{-(z-vt)^2/a^2}\,\hat{s}$$

$$-\frac{2\alpha}{sa^2}\left[e^{-(z-vt)^2/a^2} + (z-vt)^2\left(\frac{-2}{a^2}\right)e^{-(z-vt)^2/a^2}\right]\hat{z}$$

$$= \frac{2\alpha}{s^2a^2}(z-vt)\,e^{-(z-vt)^2/a^2}\,\hat{s}$$

$$-\frac{2\alpha}{sa^2}\left[1-\frac{2(z-vt)^2}{a^2}\right]e^{-(z-vt)^2/a^2}\,\hat{z}$$

Assembling these gives:

$$\vec{J} = -\frac{1}{\mu_0}\left\{\frac{\alpha}{s}\left[\frac{1}{s^2}\cancel{-\frac{2}{a^2}\left(1-\frac{2(z-vt)^2}{a^2}\right)}\right]\hat{z} + \frac{2\alpha\mu_0\epsilon_0 v^2}{sa^2}\left[1-\frac{2(z-vt)^2}{a^2}\right]\hat{z}\right.$$

$$\left. \cancel{+\frac{2\alpha}{sa^2}\left[1-\frac{2}{a^2}(z-vt)^2\right]\hat{z}} + \frac{2\alpha}{s^2a^2}(z-vt)\,\hat{s}\right\}e^{-(z-vt)^2/a^2}$$

$$= -\frac{1}{\mu_0}\left\{\frac{\alpha}{s^3} + \frac{\alpha}{sa^2}\frac{v^2}{c^2}\left[1-\frac{2(z-vt)^2}{a^2}\right]\right\}e^{-(z-vt)^2/a^2}\,\hat{z}$$

$$-\frac{1}{\mu_0}\frac{2\alpha}{s^2a^2}(z-vt)\,e^{-(z-vt)^2/a^2}\,\hat{s}$$

Continuity equation

The continuity equation is

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0.$$

Then

$$\rho = -\epsilon_0 \left\{ \nabla^2 V + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} \right\} \Rightarrow \quad \frac{\partial \rho}{\partial t} = -\epsilon_0 \left\{ \nabla^2 \frac{\partial V}{\partial t} + \frac{\partial^2}{\partial t^2}\vec{\nabla}\cdot\vec{A} \right\}$$

Also

$$\vec{\nabla}\cdot\vec{J} = -\frac{1}{\mu_0}\vec{\nabla}\cdot\left\{ \nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) \right\}.$$

$$= -\frac{1}{\mu_0}\vec{\nabla}\cdot\left(\nabla^2\vec{A}\right) + \epsilon_0\frac{\partial^2}{\partial t^2}\vec{\nabla}\cdot\vec{A} + \frac{1}{\mu_0}\nabla^2\left(\vec{\nabla}\cdot\vec{A}\right) + \epsilon_0\nabla^2\frac{\partial V}{\partial t}.$$

Then

$$\vec{\nabla}\cdot\left(\nabla^2\vec{A}\right) = \frac{\partial}{\partial x}\nabla^2 A_x + \frac{\partial}{\partial y}\nabla^2 A_y + \frac{\partial}{\partial z}\nabla^2 A_z$$

$$= \nabla^2\left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right]$$

$$= \nabla^2\left(\vec{\nabla}\cdot\vec{A}\right)$$

Adding the two terms gives

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

as expected.

Gauge Transformations.

Physical quantities are usually obtained from potentials by differentiation and this is clearly true in electromagnetism. Thus we expect that many potentials might give the same fields. Consider electrostatics. In this case $\frac{\partial \vec{A}}{\partial t} = 0$ and

$$\vec{E} = -\vec{\nabla} V$$

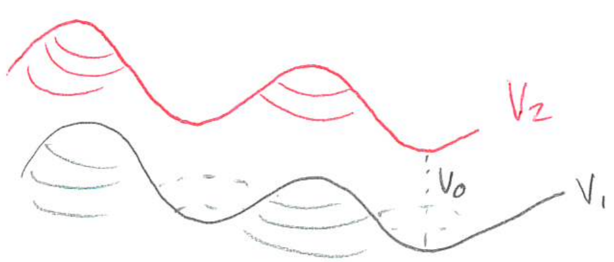
Likewise

$$\nabla^2 V = -\rho/\epsilon_0$$

We could see that if $V_1(\vec{r})$ is a solution to the Poisson equation, i.e.

$$\nabla^2 V_1 = -\rho/\epsilon_0$$

then if V_0 is an overall constant, the potential $V_2 = V_1 + V_0$ will clearly also satisfy the Poisson equation. This simply resets the "zero" of potential. The gradients do not change and therefore the fields do not change.



$\Rightarrow$ same $\vec{E}$ field

$\Rightarrow$ same ρ

3 Gauge freedom in electrostatics

The field produced by an infinite uniformly charged sheet at $z = 0$ is

$$\mathbf{E} = \begin{cases} +\dfrac{\sigma}{\epsilon_0}\hat{\mathbf{z}} & \text{if } z > 0 \\ -\dfrac{\sigma}{\epsilon_0}\hat{\mathbf{z}} & \text{if } z < 0 \end{cases}$$

where σ is the surface charge density.

a) Show that one possibility for the potential is

$$V_1 = \begin{cases} -\dfrac{\sigma}{\epsilon_0}z & \text{if } z > 0 \\ +\dfrac{\sigma}{\epsilon_0}z & \text{if } z < 0 \end{cases}$$

b) Suppose that $V_2 = V_1 + f(x, y, z)$ produces the same electric field as V_1. Determine a general expression for $f(x, y, z)$ that results in the same electric field.

Answer: a) $\vec{\nabla} V_1 = +\cancel{\dfrac{\partial V_1}{\partial x}}\hat{x} + \cancel{\dfrac{\partial V_1}{\partial y}}\hat{y} + \dfrac{\partial V_1}{\partial z}\hat{z}$

$$= \frac{\partial V_1}{\partial z}\hat{z}$$

So $\vec{\nabla} V_1 = \begin{cases} -\dfrac{\sigma}{\epsilon_0}\hat{z} & \text{if } z>0 \\ \dfrac{\sigma}{\epsilon_0}\hat{z} & \text{if } z<0 \end{cases}$

$\Rightarrow \vec{E} = -\vec{\nabla} V_1$. So this works.

b) $-\vec{\nabla} V_2 = -\vec{\nabla} V_1 \Rightarrow \cancel{-\vec{\nabla} V_1} - \vec{\nabla} f = \cancel{-\vec{\nabla} V_1}$

$\Rightarrow \vec{\nabla} f = 0$

$\Rightarrow \dfrac{\partial f}{\partial x}\hat{x} + \dfrac{\partial f}{\partial y}\hat{y} + \dfrac{\partial f}{\partial z}\hat{z} = 0$

$\Rightarrow \dfrac{\partial f}{\partial x} = 0 \quad \dfrac{\partial f}{\partial y} = 0 \quad \dfrac{\partial f}{\partial z} = 0 \quad \Rightarrow f(x,y,z) = \text{constant}$.

The shift from one potential to another that give the same field is called a gauge transformation.

$$
\begin{tikzcd}
\text{Potential } V_1 \arrow[rr, leftrightarrow, "\text{gauge transf.}"'] \arrow[d, squiggly] & & \text{Potential } V_2 \arrow[d, squiggly] \\
\text{Field } \vec{E} = -\vec{\nabla} V_1 \arrow[rr, leftrightarrow, bend right, "\text{same field}"'] & & \text{Field } \vec{E} = -\vec{\nabla} V_2
\end{tikzcd}
$$

We have shown that in electrostatics the possible gauge transformation is

$$V \longrightarrow V + \underbrace{V_0}_{\text{constant.}}$$

When vector fields are present there will be more possibilities.