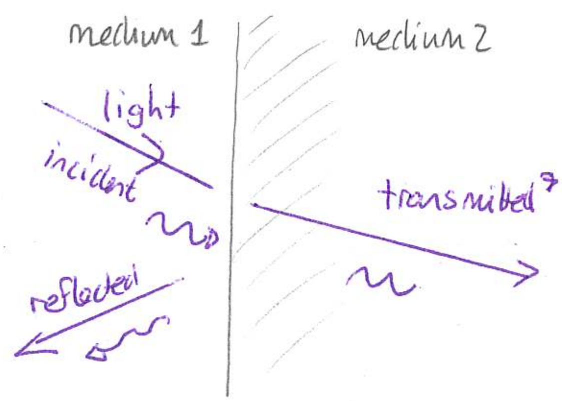
Tues: HW 5pm

Thurs: Read 9.1.4, 9.2.1, 9.2.2.

Reflection and Transmission of Waves

In a homogeneous medium the speed with which waves travel is the same at all locations. What happens when the medium is not homogeneous?

This occurs in electromagnetic theory and in optics, often when there is an abrupt boundary between the two media. We typically find that light incident from one medium is partly transmitted into the other medium and partly reflected back into the original medium.



The issue will be:

> Given a incident wave which supplies energy at a certain rate, what are the rate and direction at which energy is transmitted and what are the rate and direction at which energy is reflected?

We will analyze this question for waves on a string.. Recall that, for these waves,

$$\boxed{\begin{array}{l}\text{The rate at which energy propagates from left to right at the location } z \text{ is:} \\ \\ P = -T\left(\dfrac{\partial f}{\partial t}\right)_z \left(\dfrac{\partial f}{\partial z}\right)_z\end{array}}$$

Then for a traveling sinusoidal wave

$$f(z,t) = A\cos(kz \mp \omega t)$$

we get

$$P = \pm T\omega k\, A^2 \sin^2(kz \mp \omega t)$$

Using $\omega = vk$ we get $k = \omega/v$ and

$$P = \pm \frac{T\omega^2}{V} A^2 \sin^2(kz \mp \omega t)$$

Note that the power delivered depends on:

1) the amplitude via A^2

2) the wavespeed

So we need to track these in both media. This will be true for all types of waves.

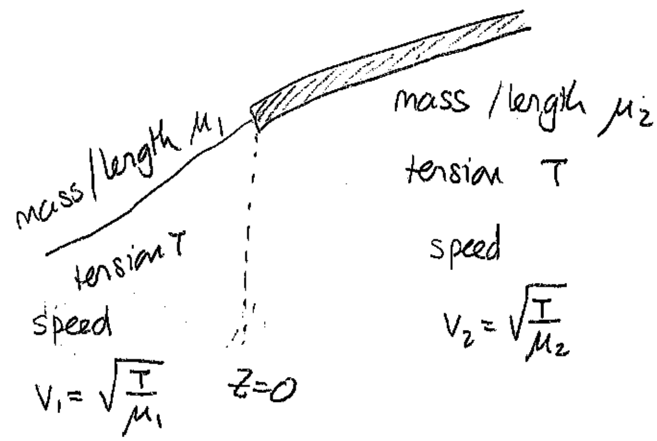
Given two media with distinct wavespeeds we need to track the amplitudes of the waves on either side of the junction. This will require matching conditions at the boundary.

Matching conditions for waves on a string

One example of waves on strings with a jump in the medium involves strings with different masses per unit length and the same tension.

Assume that the boundary occurs at $z=0$. To the left ($z<0$) the wave equation is

$$\frac{\partial^2 f_1}{\partial z^2} = \frac{1}{V_1^2}\frac{\partial^2 f_1}{\partial t^2}$$



and the solution will be denoted $f_1(z,t)$

To the right ($z>0$)

$$\frac{\partial^2 f_2}{\partial z^2} = \frac{1}{V_2^2}\frac{\partial^2 f_2}{\partial t^2}$$

with solution $f_2(z,t)$. Thus we have

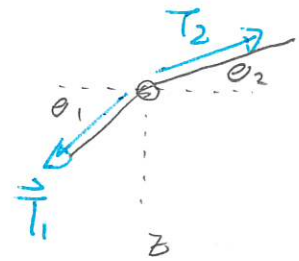
$$f(z,t) = \begin{cases} f_1(z,t) & z \leq 0 \\ f_2(z,t) & z \geq 0 \end{cases}$$

How do the two solutions match at the boundary?

If the strings are not broken then

$$\boxed{f_1(0,t) = f_2(0,t).}$$

The other condition relates to the motion of the segment at $z=0$. Here



$$\vec{F}_{net} = m\vec{a}$$

and as the segment $\to 0$, $m \to 0 \Rightarrow \vec{F}_{net} \to 0$. Thus

$$\vec{T}_1 = \vec{T}_2 \quad \Rightarrow \quad T_{1x} = -T_{2x}$$

$$T_{1y} = -T_{2y}$$

But $T_{1x} = -T_1 \cos\theta_1 \qquad T_{1y} = -T_1 \sin\theta_1$

$T_{2x} = T_2 \cos\theta_2 \qquad T_{2y} = T_2 \sin\theta_2$

gives

$$+T_1 \cos\theta_1 = T_2 \cos\theta_2$$

$$+T_1 \sin\theta_1 = T_2 \sin\theta_2$$

For small angles θ_1, θ_2 we get $\cos\theta_1 \approx 1$ and $\cos\theta_2 \approx 1$ so $T_1 = T_2$. Then

$$T_1 \sin\theta = T_2 \sin\theta_2 \Rightarrow \sin\theta_1 = \sin\theta_2 \Rightarrow \frac{\sin\theta_1}{\cos\theta_1} = \frac{\sin\theta_2}{\cos\theta_2}$$

$$\Rightarrow \tan\theta_1 \approx \tan\theta_2$$

Now $\tan\theta \approx \frac{\partial f}{\partial z}$ Thus we get

$$\left.\frac{\partial f_1}{\partial z}\right|_{z=0} = \left.\frac{\partial f_2}{\partial z}\right|_{z=0}$$

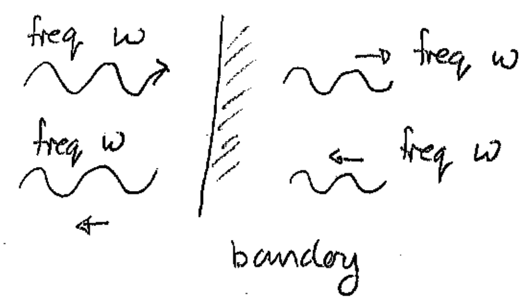
We thus reach:

Consider a string whose only discontinuity is a jump in the mass per unit length at $z=0$. Then let $f_1(z,t)$ satisfy the wave equation for $z<0$ and $f_2(z,t)$ for $z>0$. At the boundary the matching conditions are:

$$f_1(0,t) = f_2(0,t)$$

$$\left.\frac{\partial f_1}{\partial z}\right|_{z=0} = \left.\frac{\partial f_2}{\partial z}\right|_{z=0}.$$

These apply equally well to complex representations of waves. We now consider the case where the frequency of either wave is the same. So on the left waves traveling left and right both have frequency ω_1; on the right frequency ω_2. At the boundary the two frequencies must match. So we consider a situation where there is only one frequency present. Then the wavenumbers are:



On left : $k_1 = \omega/v_1$

On right: $k_2 = \omega/v_2$

and these will be different.

Then the general solution on the left is

$$f_1(z,t) = \underbrace{A_1 e^{i(k_1 z - \omega t)}}_{\text{travels right}} + \underbrace{B_1 e^{-i(k_1 z + \omega t)}}_{\text{travels left}}$$

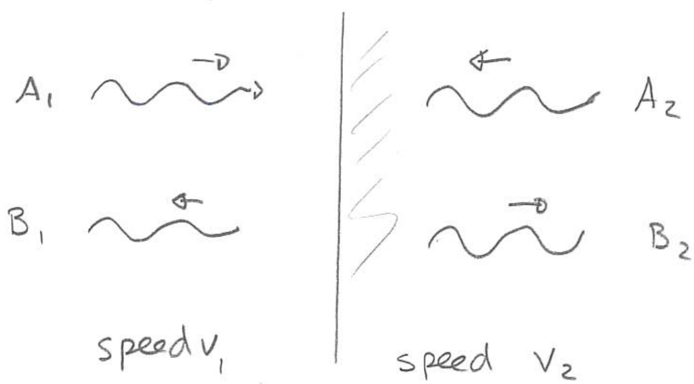
$\equiv$ incident from left $\qquad\qquad$ $\equiv$ reflected/transmitted left

$$f_2(z,t) = \underbrace{B_2 e^{i(k_2 z - \omega t)}}_{\text{travels } \cancel{\text{left}} \text{ right}} + \underbrace{A_1 e^{-i(k_2 z + \omega t)}}_{\text{travels } \cancel{\text{right}} \text{ left}}$$

reflected/transmitted right $\qquad\qquad$ $\equiv$ incident from right.

Schematically we can illustrate this as →

We then need to relate the A, B coefficients.



$\omega = k_1 v_1 \qquad\qquad \omega = k_2 v_2$

1 Boundary matching conditions for a string

a) Apply the matching conditions to relate A_1, A_2, B_1 and B_2.

b) Express the matching conditions using matrices:

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

c) The inverse of a matrix that has non-zero determinant is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Use this and the previous result to determine expressions for B_1 and B_2 in terms of A_1 and A_2.

Answer: a) $f_1(0,t) = f_2(0,t)$

$$A_1 e^{-i\omega t} + B_1 e^{-i\omega t} = B_2 e^{-i\omega t} + A_2 e^{-i\omega t}$$

$$\Rightarrow \quad A_1 + B_1 = B_2 + A_2$$

$$\left.\frac{\partial f_1}{\partial z}\right|_{z=0} = \left.\frac{\partial f_2}{\partial z}\right|_{z=0}$$

$$\Rightarrow \quad iA_1 k_1 e^{-i\omega t} - ik_1 B_1 e^{-i\omega t} = ik_2 B_2 e^{-i\omega t} - ik_2 A_2 e^{-i\omega t}$$

$$\Rightarrow \quad A_1 k_1 - B_1 k_1 = k_2 B_2 - k_2 A_2$$

So

$$A_1 + B_1 = B_2 + A_2$$

$$A_1 k_1 - B_1 k_1 = k_2 B_2 - k_2 A_2$$

b) $$B_1 - B_2 = -A_1 + A_2$$

$$-B_1 k_1 - B_2 k_2 = -k_2 A_2 - k_1 A_1 \quad \Rightarrow \quad B_1 k_1 + B_2 k_2 = k_1 A_1 + k_2 A_2$$

Then

$$\begin{pmatrix} 1 & -1 \\ k_1 & k_2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ k_1 & k_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

c) We need to invert

$$\begin{pmatrix} 1 & -1 \\ k_1 & k_2 \end{pmatrix}$$

Then $\begin{pmatrix} 1 & -1 \\ k_1 & k_2 \end{pmatrix}^{-1} = \dfrac{1}{k_2 + k_1} \begin{pmatrix} k_2 & 1 \\ -k_1 & 1 \end{pmatrix}$

Thus

$$\cancel{\begin{pmatrix} 1 & -1 \\ k_1 & k_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 \\ k_1 & k_2 \end{pmatrix}} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \frac{1}{k_1 + k_2} \begin{pmatrix} k_2 & 1 \\ -k_1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ k_1 & k_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \frac{1}{k_1 + k_2} \begin{pmatrix} -k_2 + k_1 & +2k_2 \\ +2k_1 & k_2 - k_1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

This gives:

$$\boxed{\begin{aligned} f_1(z,t) &= A_1 e^{i(k_1 z - \omega t)} + B_1 e^{-i(k_1 z + \omega t)} \\ f_2(z,t) &= B_2 e^{i(k_2 z - \omega t)} + A_2 e^{-i(k_2 z + \omega t)} \\ &\text{with matching conditions giving} \\ B_1 &= \frac{k_1 - k_2}{k_2 + k_1} A_1 + \frac{2k_2}{k_2 + k_1} A_2 \\ B_2 &= \frac{2k_1}{k_2 + k_1} A_1 + \frac{k_2 - k_1}{k_2 + k_1} A_2 \end{aligned}}$$

Waves incident from the left.

If waves are incident from the left then $A_2 = 0$. So we get

$$f_1(z,t) = \underbrace{A_1 e^{i(k_1 z - \omega t)}}_{\text{incident}} + \underbrace{B_1 e^{-i(k_1 z + \omega t)}}_{\text{reflected}}$$

$$f_2(z,t) = \underbrace{B_2 e^{i(k_2 z - \omega t)}}_{\text{transmitted.}}$$

Then $$B_1 = \frac{k_1 - k_2}{k_1 + k_2} A_1$$

$$B_2 = \frac{2k_1}{k_1 + k_2} A_1$$

2 Reflection and transmission of waves along a string

Consider waves on a string with a discontinuity at $z = 0$. Suppose that the waves are incident from the left with amplitude $A > 0$.

a) Determine conditions under which the reflected waves have "negative" amplitude. In such circumstances, what would the negative amplitude imply?

b) Does the transmitted wave ever have a "negative" amplitude?

c) Determine conditions under which the amplitude of the transmitted wave is greater than that of the incident wave.

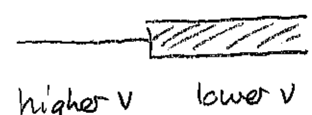
d) Can the reflected wave ever have a greater amplitude than the transmitted wave?

a) Reflected amplitude when $B_1 < 0 \Rightarrow k_2 > k_1$

But $k = \omega / v \Rightarrow \omega / v_2 > \dfrac{\omega}{v_1}$

$\Rightarrow v_1 > v_2$

The reflected wave will be inverted if the medium encountered by the incident wave has lower speed



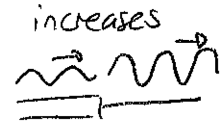
reflected is inverted.

b) No $k_1 > 0 \Rightarrow B_2 > 0$

c) Need $B_2 > A_1 \Rightarrow \dfrac{2k_1}{k_1 + k_2} > 1 \Rightarrow 2k_1 > k_1 + k_2 \Rightarrow k_1 > k_2$

$\Rightarrow V_2 > V_1$

when it travels faster in second medium amplitude increases



d) requires $|B_1| > |A| \Leftrightarrow \dfrac{k_1 - k_2}{k_1 + k_2} > 1 \Rightarrow -k_2 > k_2 \Rightarrow 0 > k_2$ impossible

or $\dfrac{k_1 - k_2}{k_1 + k_2} < -1 \Rightarrow k_1 < 0$ impossible

We analyze the situation in terms of the energy transmitted + reflected. The incident wave transports energy with power

$$P_{incident} = \frac{T\omega^2}{V_1} A_1^2 \sin^2(k_1 z - \omega t)$$

The reflected wave transports energy with power

$$P_{refl} = \frac{T\omega^2}{V_1} B_1^2 \sin^2(k_1 z + \omega t)$$

The transmitted wave transports energy with power

$$P_{trans} = \frac{T\omega^2}{\underline{\underline{V_2}}} B_2 \sin^2(k_2 z - \omega t)$$

We form the ratios.

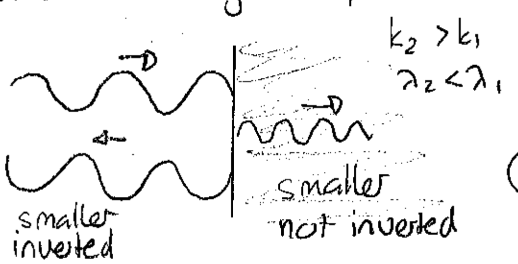
$$\frac{P_{refl}}{P_{inc}} = \left(\frac{B_1}{A_1}\right)^2 = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \leq 1$$

$$\frac{P_{trans}}{P_{inc}} = \left(\frac{B_2}{A_1}\right)^2 \frac{V_1}{V_2} = \left(\frac{2k_1}{k_1+k_2}\right)^2 \left(\frac{k_2}{k_1}\right) = \frac{4k_1 k_2}{(k_1+k_2)^2} \leq 1$$

We can see that each ratio is less than or equal to 1 and the two ratios add.

If $v_2 < v_1$

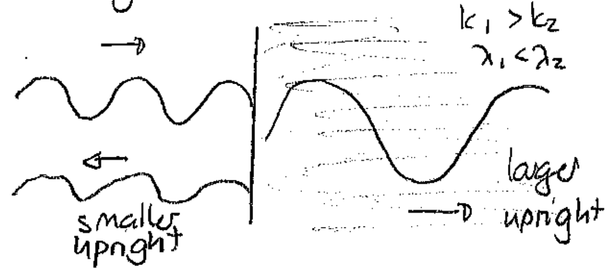
lower into higher impedance



$$\frac{P_{refl}}{P_{inc}} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$

If $V_1 < V_2$

higher into lower impedance.



$$\frac{P_{trans}}{P_{inc}} = \frac{4k_1 k_2}{(k_1 + k_2)^2}$$