Phys 312 2021

Lecture 11

Tues: HW 9 by 5pm

Read. 9.1.3, 9.1.4

Classical Waves: One Dimension

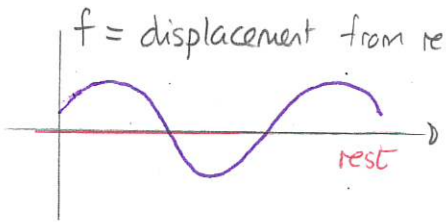
When we consider Maxwell's equations in a vacuum, we will find that these can be manipulated so as to yield equations that describe waves. We will begin by reviewing generic classical wave equations, first in one dimension.

A wave in one dimension describes a quantity f whose value varies as a function of

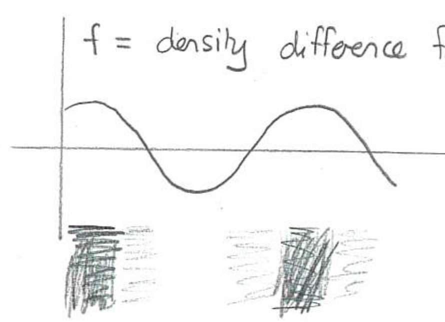
1) a single position variable, z
2) a time variable t

In this case the "one dimension" is z. Snapshots at one instant of examples are

String



Sound



The dynamics of these situations is addressed by applying relevant physics such as Newton's Laws or Thermodynamics.

What often emerges is the classical one dimensional wave equation

$$\boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

where v is a constant with units of m/s. This is called the wave speed.

A solution to the wave equation is a function $f(z,t)$ that satisfies the equation at all locations and times. The theory of differential equations provides a general solution.

> The general solution to
>
> $$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$
>
> is
>
> $$f(z,t) = g(z-vt) + h(z+vt)$$
>
> where $g(u)$ and $h(u)$ are any differentiable functions of a single variable

<u>Proof</u> Consider the case $f(z,t) = g(z-vt)$. Then

$$\frac{\partial f}{\partial z} = \left.\frac{dg}{du}\right|_{z-vt}\frac{\partial u}{\partial z} = \left.\frac{dg}{du}\right|_{z-vt}$$

$$\frac{\partial^2 f}{\partial z^2} = \left.\frac{d}{du}\left(\frac{dg}{du}\right)\right|_{z-vt}\frac{\partial u}{\partial z} = \left.\frac{d^2 g}{du^2}\right|_{z-vt}.$$

Similarly

$$\frac{\partial f}{\partial t} = \left.\frac{dg}{du}\right|_{z-vt} \underbrace{\frac{\partial u}{\partial t}}_{-v} = -v \left.\frac{dg}{du}\right|_{z-vt}$$

$$\frac{\partial^2 f}{\partial t^2} = -v \left.\frac{d}{du}\left(\frac{dg}{du}\right)\right|_{z-vt} \underbrace{\frac{\partial u}{\partial t}}_{-v}$$

$$= v^2 \left.\frac{d^2 g}{du^2}\right|_{z-vt}$$

$$= v^2 \frac{\partial^2 f}{\partial z^2} \quad \Rightarrow \quad \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

A similar derivation shows that $h(z+vt)$ is also a solution. Finally the equation is linear and thus the sum of any two solutions is also a solution ∎

1 One dimensional wave equation solutions

The one dimensional wave equation is

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}.$$ Assume $v > 0$.

a) Show that

$$f(z,t) = A\sin(kz - \omega t)$$

where k, ω and A are constants, is a solution provided that k, ω and v satisfy a particular relationship.

b) By considering the location of the maximum of

$$f(z,t) = Be^{-(z-vt)^2/a^2}$$

describe the direction of propagation and the speed with which the disturbance travels.

c) By considering the location of the maximum of

$$f(z,t) = Be^{-(z+vt)^2/a^2}$$

describe the direction of propagation and the speed with which the disturbance travels.

Answer: a) $\frac{\partial f}{\partial z} = kA\cos(kz-\omega t)$ $\qquad \frac{\partial f}{\partial t} = -\omega A\cos(kz-\omega t)$

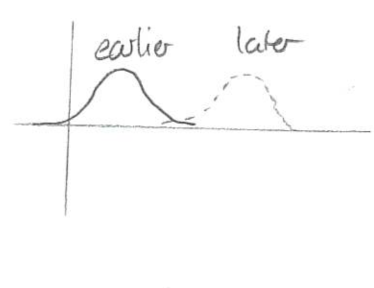
$\frac{\partial^2 f}{\partial z^2} = -k^2 A\sin(kz-\omega t)$ $\qquad \frac{\partial^2 f}{\partial t^2} = -\omega(-\omega)(-A\sin(kz-\omega t))$

$= -\omega^2 A\sin(kz-\omega t)$

$\frac{\partial^2 f}{\partial z^2} = \frac{k^2}{\omega^2}\frac{\partial^2 f}{\partial t^2} \Rightarrow$ $\quad v^2 = \omega^2/k^2 \quad \Rightarrow \boxed{v = \omega/k}$

b) Max when exponent is zero $\quad z - vt = 0 \quad \Rightarrow \quad z = vt$

max travels speed v to right



c) Max when $\quad z + vt = 0 \quad \Rightarrow z = -vt$

travels speed v to left.

In general we can interpret the two parts of the solutions as parts traveling in opposite directions. These can exist simultaneously and form superpositions. The part $g(z - vt)$ travels with velocity v and the part $h(z+vt)$ travels with velocity $-v$. We can see these by noting that if $z \mp vt = a \equiv$ constant then the function returns a constant value. This requires $z = \pm vt + a$ which means that these solutions travel with velocity $\pm v$.

Sinusoidal Waves

A basic class of solutions to the wave equation consists of sinusoidal functions.

> A sinusoidal wave has form
>
> $$f(z,t) = A\cos\left[k(z-vt)+\delta\right] = A\cos\left[kz-\omega t+\delta\right]$$
>
> where A, k, δ are constants and ω satisfies the dispersion relation:
>
> $$\omega = kv$$

The constants have the following interpretations:

1) Amplitude A = max displacement of the disturbance

2) Wavenumber k describes how the wave repeats itself spatially.

To be specific if $z' = z + 2\pi/k$ then

$$f(z',t) = f(z,t)$$

This is the shortest non-zero increment, $\Delta z = z' - z$, such that this occurs.

This increment is called the wavelength, λ. So

$$\lambda = 2\pi/k \Rightarrow k = \frac{2\pi}{\lambda}$$

3) <u>Angular frequency</u>, ω describes how the wave repeats itself in time. Let T be the shortest time s.t. $f(z, t+T) = f(z,t)$

Then:

$$\cos(kz - \omega t + \delta - \omega T) = \cos(kz - \omega t + \delta)$$

This requires $\omega T = 2\pi \Rightarrow \omega = \frac{2\pi}{T}$

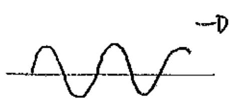
With regular frequency defined as $f = \frac{1}{T}$ we get $\omega = 2\pi f$.

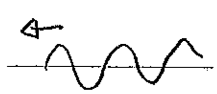
4) <u>Phase</u>, δ, establishes the displacement and transverse velocity at $z=0$ and $t=0$.

By the previous exercise and general result $f(z,t) = A\cos(kz - \omega t + \delta)$ represents a wave traveling with velocity $v = \omega/k$. Conventionally ω and k are positive and thus this represents a wave traveling right. A wave traveling left will be represented by $f(z,t) = A\cos(kz + \omega t + \delta)$.

Thus:

For $\omega, k > 0$ waves traveling along $+\hat{z}$ and $-\hat{z}$ are:

along $+\hat{z}$  $f(z,t) = A\cos(kz - \omega t + \delta)$

along $-\hat{z}$  $f(z,t) = A\cos(kz + \omega t + \delta)$

Note that these waves have a single frequency and wavenumber and extend for all z, i.e. $-\infty \le z \le \infty$. In this sense they are an idealization.

Complex Representation of Waves

Representing sinusoidal waves in terms of trigonometric functions is less convenient than representing them in terms of complex exponentials. Recall that complex numbers have the form:

$$z = u + iv$$

where u, v are real. Then the real part of z is

$$\mathrm{Re}[z] = u$$

and the imaginary part of z is

$$\mathrm{Im}[z] = v$$

An important definition is the complex exponential. For any real θ,

$$\boxed{e^{i\theta} := \cos\theta + i\sin\theta}$$

Now if A is real we consider

$$\tilde{f}(z,t) = Ae^{i(kz-\omega t+\delta)}$$

Then

$$\mathrm{Re}[\tilde{f}(z,t)] = A\cos(kz-\omega t+\delta) = f(z,t)$$

So the real part of $\tilde{f}(z,t)$ is a sinusoidal wave. We can thus use $\tilde{f}(z,t)$ to represent a wave with the understanding that the physical displacement is represented by $\mathrm{Re}[\tilde{f}(z,t)]$. When there are operations involving more than one wave, it is usually more convenient to use the complex representations to do the operations, and eventually extract the real part.

Note that

$$\tilde{f}(z,t) = Ae^{i\delta} e^{i(kz-\omega t)}$$

and if we define a complex amplitude $\tilde{A} = Ae^{i\delta}$ then

$$\tilde{f}(z,t) = \tilde{A} e^{i(kz-\omega t)}$$

so we have the following scheme:

Real wave equation

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Complex wave equation

$$\frac{\partial^2 \tilde{f}}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \tilde{f}}{\partial t^2}$$

Real sinusoidal solns:

travel along $+\hat{z}$ $\Big\}$ $f(z,t) = A\cos(kz-\omega t+\delta)$

travel along $-\hat{z}$ $\Big\}$ $f(z,t) = A\cos(kz+\omega t+\delta)$

Complex exponential solutions

along $+\hat{z}$ $\quad \tilde{f}(z,t) = \tilde{A}\, e^{i(kz-\omega t)}$

along $-\hat{z}$ $\quad \tilde{f}(z,t) = \tilde{A}\, e^{i(kz+\omega t}$

Do math here.

same solution

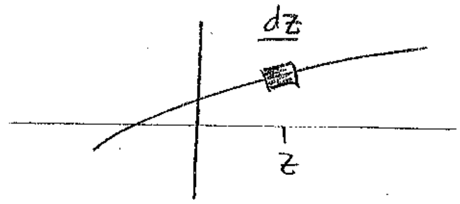
$$f(z,t) = Re\left[f(z,t)\right]$$

When finished translate here to get physical solutions.

Energy in waves

An important consideration for waves are their energy content.

Consider a wave on a string with tension T and mass per unit length μ. Then the kinetic energy of the shaded portion is



$$dK = \frac{1}{2}\underbrace{\mu dz}_{dm}\underbrace{\left(\frac{\partial f}{\partial t}\right)^2}_{v^2_{transv.}}$$

The kinetic energy for $a \leq z \leq b$ is

$$K = \frac{1}{2}\int_a^b \mu\left(\frac{\partial f}{\partial t}\right)^2 dz$$

What would a possible potential energy be? We show that

$$U = \frac{1}{2}\int_a^b T\left(\frac{\partial f}{\partial z}\right)^2 dz$$

gives a suitable potential energy. Then

In the region $a \leq z \leq b$ the total energy stored in the wave is

$$E = \frac{1}{2}\int\left[\mu\left(\frac{\partial f}{\partial t}\right)^2 + T\left(\frac{\partial f}{\partial z}\right)^2\right]dz$$

and

$$\frac{dE}{dt} = \underbrace{-T\left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial f}{\partial z}\right)_a}_{\text{rate at which energy enters at } a} + \underbrace{T\left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial f}{\partial z}\right)_b}_{\text{rate at which energy leaves at } b}$$

Proof:

$$\frac{dE}{dt} = \frac{1}{2}\int \mu \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right)^2 dz + \frac{1}{2}\int T \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial z}\right)^2 dz$$

$$= \frac{1}{2}\int \mu\, 2\, \frac{\partial f}{\partial t}\left(\frac{\partial^2 f}{\partial t^2}\right) dz + \frac{1}{2}\int_a^b 2T\, \underbrace{\frac{\partial f}{\partial z}}_{u}\, \underbrace{\frac{\partial^2 f}{\partial t \partial z}\, dz}_{dv}$$

int by parts

$$= \int \mu \frac{\partial f}{\partial t}\frac{\partial^2 f}{\partial t^2} dz + T\left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial f}{\partial t}\right)\Big|_a^b - T\int \frac{\partial^2 f}{\partial z^2}\frac{\partial f}{\partial t} dz$$

$$= \underbrace{\int \mu \frac{\partial f}{\partial t}\left[\frac{\partial^2 f}{\partial t^2} - \underbrace{\frac{T}{\mu}}_{=v^2}\frac{\partial^2 f}{\partial z^2}\right] dz}_{=0 \text{ by wave eqn.}} + T\left(\frac{\partial f}{\partial z}\right)_b\left(\frac{\partial f}{\partial t}\right)_b - T\left(\frac{\partial f}{\partial z}\right)_a\left(\frac{\partial f}{\partial t}\right)_a$$

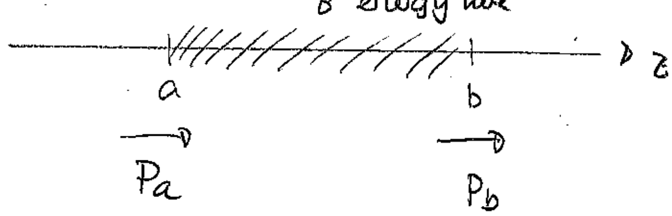
∎

We interpret the product of the derivatives as the rate at which energy is delivered from left to right. So we define this as the power delivered

$$P = -T\left(\frac{\partial f}{\partial t}\right)_a\left(\frac{\partial f}{\partial z}\right)_a$$

$$E = \frac{1}{2}\int_a^b \left\{\mu\left(\frac{\partial f}{\partial t}\right)^2 + T\left(\frac{\partial f}{\partial z}\right)^2\right\} dz$$

energy here



describes energy flowing this way

describes energy flowing this way

2 Energy and power for one dimensional waves

Consider two sinusoidal waves

$$f_1(z,t) = A\cos(kz - \omega t)$$
$$f_2(z,t) = A\cos(kz + \omega t)$$

where $k > 0$ and $\omega > 0$. Assume that these are waves on a string, for which $v^2 = T/\mu$.

a) Determine the energy stored over the span of a single wavelength for each wave.

b) Determine the power delivered at any point. Does it depend on the location and the time? What does this power indicate about the direction in which each wave transports energy?

Answer: a)

$$E = \frac{1}{2}\int_{z_0}^{z_0+\lambda}\left\{\mu\left(\frac{\partial f}{\partial t}\right)^2 + T\left(\frac{\partial f}{\partial z}\right)^2\right\}dz$$

$$= \frac{1}{2}\int_{z_0}^{z_0+\lambda}\Big\{\mu(\mp\omega)^2 A^2\cos^2(kz-\omega t) + T k^2 A^2\cos^2(kz-\omega t)\Big\}dz$$

But $T = v^2\mu = (\omega^2/k^2)\mu$ gives $\qquad v = \omega/k.$

$$E = \frac{\mu}{\cancel{2}}\omega^2 \cancel{2}\int_{z_0}^{z_0+\lambda} A^2\cos^2(kz-\omega t)\,dz.$$

$$= \frac{\mu\omega^2 A^2}{2}\lambda = \frac{\mu\omega^2}{\cancel{2}}A^2\frac{\cancel{2}\pi}{k} \qquad \lambda = 2\pi/k.$$

$$= \mu\omega v\pi A^2$$

b) $$P = -T\left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial f}{\partial z}\right) = +T(\mp\omega)A\sin(kz-\omega t)(-kA\sin(kz-\omega t))$$

$$= \mp T\omega k A^2\sin^2(kz-\omega t)$$

For f_1 we get

$$P = T\omega k\, A^2 \sin^2(kz-\omega t) \quad > 0$$

power delivered $\longrightarrow$

For f_2 we get

$$P = -T\omega k\, A^2 \sin^2(kz-\omega t) \quad < 0$$

$\longleftarrow$ power delivered

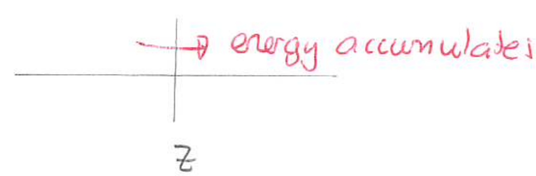
Both depend on location and time. But for the wave with $kz-\omega t$, the power is delivered to the right while for the other it is delivered to the left.

Note $T\omega k = v^2 \mu \omega k$ and $k = \omega/v$ gives

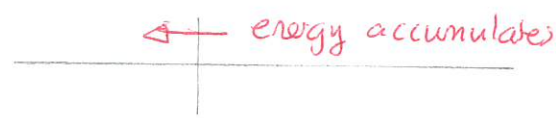
$$T\omega k = v \mu \omega^2$$

Thus

$A\cos(kz-\omega t)$ gives $P = \mu v \omega^2 A^2 \sin^2(kz-\omega t)$



$A\cos(kz+\omega t)$ " $P = -\mu v \omega^2 A^2 \sin^2(kz-\omega t)$



Note: Power delivered depends on
- amplitude
- wave speed
- mass/unit length