Thurs: HW by 5pm

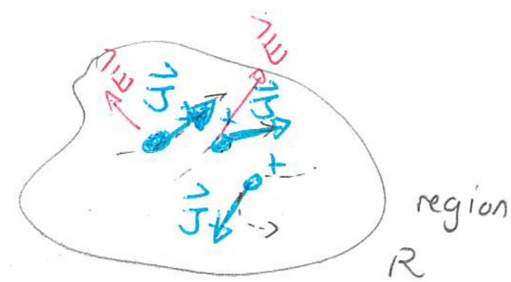
Thurs: Read 8.2.3

Tues: Exam covers all up to today's class

Energy conservation in electromagnetism

We consider situations in which source charges and currents produce electric and magnetic fields which act on these charges. We were then able to show that the rate of change of kinetic energy of the matter within a region is

$$\frac{dK_{region}}{dt} = \int_{region} \vec{J}\cdot\vec{E}\, d\tau'$$



where $\vec{J}(\vec{r}') =$ current density within region

$\vec{E}(\vec{r}') =$ electric field within region (produced by all sources within and beyond the region).

Then the total potential energy within the region is:

$$U_{region} = \frac{\epsilon_0}{2}\int_{region} \vec{E}\cdot\vec{E}\, d\tau' + \frac{1}{2\mu_0}\int_{region} \vec{B}\cdot\vec{B}\, d\tau'$$

Using Maxwell's equations and vector calculus we can prove that

$$\boxed{\frac{d}{dt}\left(U_{region} + K_{region}\right) = -\oint_{\text{surface of region}} \vec{S}\cdot d\vec{a}}$$

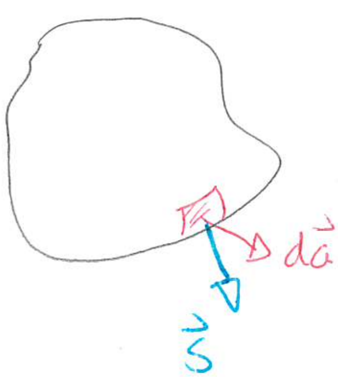
where

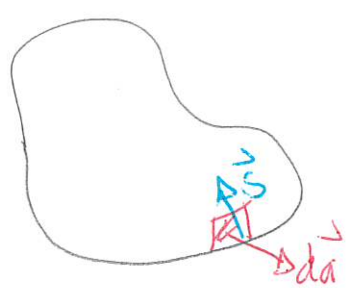
$$\boxed{\vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)}$$

is the Poynting vector. This general theorem then means that the Poynting vector describes:

1) the rate at which energy leaves the surface w.r.t. time
2) " " " " " " " " w.r.t area
3) the direction in which the energy flows.

For a closed surface this will make sense if the area vector $d\vec{a}$ is directed outward





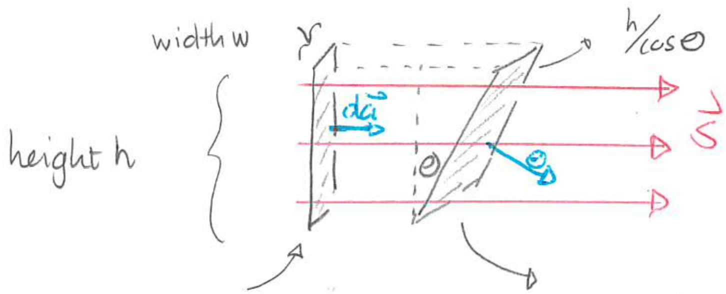
$\vec{S}\cdot d\vec{a} > 0$ $\qquad$ $\vec{S}\cdot d\vec{a} < 0$

energy leaves interior $\qquad$ energy enters interior

$\Rightarrow$ $\vec{S}$ points in direction of energy flow

The energy flow rate does require a surface over which to integrate.

Consider the examples



Here $\vec{S} \cdot d\vec{a} = Shw$ Here $\vec{S} \cdot d\vec{a} = S \frac{h}{\cos\theta} \cdot w \cos\theta = Shw.$

Thus

$\vec{S}$ quantifies the energy flow per unit area through a surface perpendicular to $\vec{S}$

Finally the units of this are attained via:

$$\left.\begin{aligned} E &\sim N/C \\ B &\sim T = \frac{N}{A \cdot m} \\ \mu_0 &\sim N/A^2 \end{aligned}\right\} \qquad \frac{1}{\mu_0} EB \sim \frac{N}{C} \frac{\cancel{N}}{A \cdot m} \cdot \frac{A^{\cancel{2}}}{\cancel{N}}$$

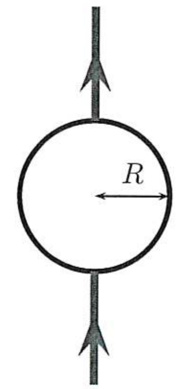
$$= \frac{N\, \cancel{C}/s}{\cancel{C}\, m} = \frac{Nm}{m^2 s}$$

$$= \frac{J}{s \cdot m^2}$$

and these are units of energy per second per meter squared.

1 Poynting vector and energy flow

A small spherical shell with radius R carries charge $q > 0$ which is uniformly distributed. Two wires carry constant currents in the indicated directions. The wires are very long compared to the sphere radius.



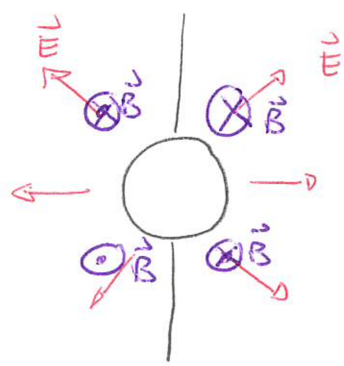
a) Sketch the electric and magnetic fields produced by charge and current distributions.

b) Determine the Poynting vector associated with these fields.

c) In which general sense does energy flow in this situation. Does this match what would be expected in terms of any mechanical work done?

d) Determine the rate at which energy flows through the entire horizontal plane through the middle of the sphere.

Answer: a)



b) Using Gauss' Law

$$\vec{E} = \begin{cases} 0 & r < R \\ \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} & r > R \end{cases}$$

Using Ampère's Law

$$\vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$

So

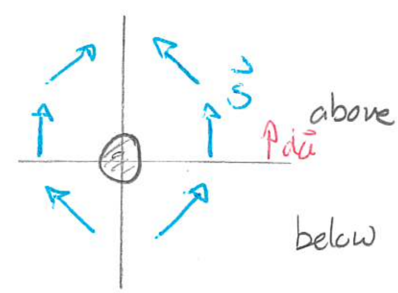
$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = 0 \quad \text{inside}$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$$= \frac{1}{\mu_0}\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\frac{\mu_0 I}{2\pi s}\underbrace{\hat{r}\times\hat{\phi}}_{-\hat{\theta}}$$

But $s = r\sin\theta$ gives

$$\vec{S} = -\frac{qI}{8\pi^2\epsilon_0 r^3\sin\theta}\hat{\theta}$$



c) It tends to flow from below to above

We can see that in the wire above $\vec{J}\cdot\vec{E} > 0$ and below $\vec{J}\cdot\vec{E} < 0$. So the kinetic energy in the wire above increase and it decreases in the wire below. The flow of energy must be from below to above.

d) Integrate over the surface from

$$\left.\begin{array}{r} R \le r' < \infty \\ \theta' = \pi/2 \\ 0 \le \phi' \le 2\pi \end{array}\right\} d\vec{a} = r'\sin\theta' dr' d\phi' (-\hat{\theta})$$

$$= -r' dr' d\phi' \hat{\theta}$$

So $\quad \vec{S}\cdot d\vec{a} = \dfrac{qI}{8\pi^2\epsilon_0 r'^2} dr' d\phi'$

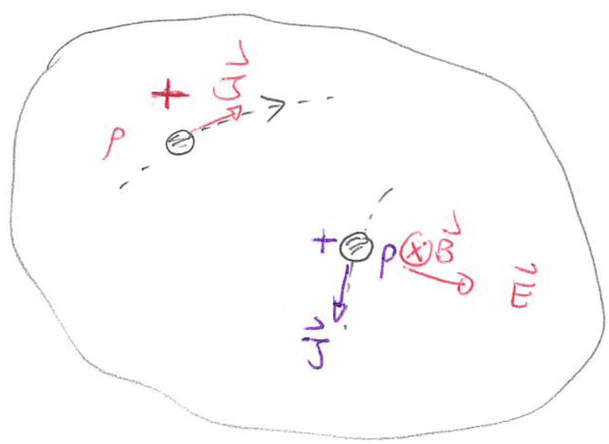
$$\oint \vec{S}\cdot d\vec{a} = \frac{qI}{8\pi^2\epsilon_0}\int_R^\infty dr' \frac{1}{r'^2}\int_0^{2\pi} d\phi' = \frac{qI}{4\pi\epsilon_0 R} > 0$$

Momentum in electromagnetism

Since electric and magnetic fields exert force on charges, we expect that the will be able to change the momentum of the charges on which they act. We will therefore expect a version of the energy conservation situation.

Fields carry momentum $\rightsquigarrow$ Momentum of fields and matter is conserved

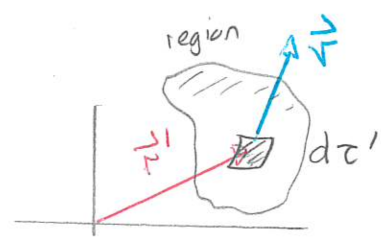
We again assume a "universe" in which charged matter produces and responds to fields.



We consider the force exerted by the fields on some small portion of charge located at $\vec{r}\,'$ and occupying volume $d\tau'$. The force exerted on these is

$$\vec{F} = \rho(\vec{r}\,')d\tau'\left[\vec{E} + \vec{v}(\vec{r}\,')\times\vec{B}\right]$$

velocity



If we extend this to a finite region R we get that the net force on the matter in this region is:

$$\vec{F} = \int_{\text{Region}} \rho\left[\vec{E} + \vec{v}\times\vec{B}\right]d\tau'$$

$$\Rightarrow \quad \vec{F} = \int_{\text{region}} \left[\rho \vec{E} + \vec{J} \times \vec{B}\right] d\tau'$$

We now use Maxwell's equations to eliminate ρ and $\vec{J}$. Then

$$\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}$$

$$\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

This gives:

$$\vec{F} = \int_{\text{region}} \left\{ \epsilon_0 (\vec{\nabla} \cdot \vec{E}) \vec{E} + \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \times \vec{B} \right\} d\tau'$$

Now

$$\frac{\partial}{\partial t} (\vec{E} \times \vec{B}) = \frac{\partial \vec{E}}{\partial t} \times \vec{B} + \vec{E} \times \frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \quad \frac{\partial \vec{E}}{\partial t} \times \vec{B} = \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) - \vec{E} \times \frac{\partial \vec{B}}{\partial t}$$

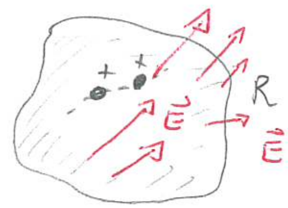
$$= \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) + \vec{E} \times (\vec{\nabla} \times \vec{E})$$

This gives:

$$\vec{F} = -\epsilon_0 \frac{\partial}{\partial t} \int_{\text{region}} (\vec{E} \times \vec{B}) \, d\tau + \int_{\text{region}} \left\{ \epsilon_0 (\vec{\nabla} \cdot \vec{E}) \vec{E} + \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) \times \vec{B} - \vec{E} \times (\vec{\nabla} \times \vec{E}) \right\} d\tau'$$

Thus in general we have; using the additional fact that $\vec{\nabla}\cdot\vec{B}=0$,

Consider a system of charged particles within a region R. Let $\vec{E}$ and $\vec{B}$ be the fields produced by these particles and any others outside the region. Then the net force on the particles in this region is



$$\vec{F} = -\epsilon_0 \frac{d}{dt}\int \vec{E}\times\vec{B}\,d\tau + \int\Big[\epsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E}) + \frac{1}{\mu_0}(\vec{\nabla}\cdot\vec{B})\vec{B} - \frac{1}{\mu_0}\vec{B}\times(\vec{\nabla}\times\vec{B})\Big]d\tau.$$

Separately from classical mechanics the mechanical momentum satisfies

$$\vec{F} = \frac{d\vec{P}_{mech}}{dt}$$

and thus

$$\frac{d\vec{P}_{mech}}{dt} = -\epsilon_0 \frac{d}{dt}\int_{\text{region}} (\vec{E}\times\vec{B})\,d\tau + \int_{\text{region}}\Big[\epsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E}) + \frac{1}{\mu_0}(\vec{\nabla}\cdot\vec{B})\vec{B} - \frac{1}{\mu_0}\vec{B}\times(\vec{\nabla}\times\vec{B})\Big]d\tau.$$

Note that $\vec{P}_{mech}$ is the entire mechanical momentum is the region. We will consider this in general. However, first we consider the special case where there are no magnetic fields. This will allow us to interpret the rightmost integral.

Forces with no magnetic fields.

Without any magnetic field, $\vec{B}=0$ and $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0$. Then we get:

$$\frac{d\vec{P}_{mech}}{dt} = \int_{region} \epsilon_0 (\vec{\nabla}\cdot\vec{E})\vec{E}\, d\tau$$

and we aim to express the integrand in terms of a gradient. We can work component by component. So with

$$\vec{P}_{mech} = P_{mech\,x}\,\hat{x} + P_{mech\,y}\,\hat{y} + P_{mech\,z}\,\hat{z}$$

we get

$$\frac{dP_{mech\,x}}{dt} = \int \epsilon_0 (\vec{\nabla}\cdot\vec{E})\, E_x\, d\tau$$

$$\frac{dP_{mech\,y}}{dt} = \int \epsilon_0 (\vec{\nabla}\cdot\vec{E})\, E_y\, d\tau.$$

$$\vdots$$

We now show that

For an electrostatic field

$$(\vec{\nabla}\cdot\vec{E})E_x = \frac{\partial}{\partial x}\left(E_x^2 - \frac{1}{2}\vec{E}\cdot\vec{E}\right) + \frac{\partial}{\partial y}(E_y E_x) + \frac{\partial}{\partial z}(E_z E_x)$$

Proof:

$$\left(\vec{\nabla}\cdot\vec{E}\right)E_x = \frac{\partial E_x}{\partial x}E_x + \frac{\partial E_y}{\partial y}E_x + \frac{\partial E_z}{\partial z}E_x$$

$$= \frac{1}{2}\frac{\partial}{\partial x}(E_x^2) + \frac{\partial}{\partial y}(E_xE_y) - E_y\frac{\partial E_x}{\partial y} + \frac{\partial}{\partial z}(E_zE_x) - E_z\frac{\partial E_x}{\partial z}.$$

Then $\vec{\nabla}\times\vec{E} = 0 \Rightarrow$

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}$$

$$\frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$

So

$$\left(\vec{\nabla}\cdot\vec{E}\right)E_x = \frac{1}{2}\frac{\partial}{\partial x}(E_x^2) + \frac{\partial}{\partial y}(E_yE_x) - E_y\frac{\partial E_y}{\partial x} + \frac{\partial}{\partial z}(E_zE_x) - E_z\frac{\partial E_z}{\partial x}$$

$$= \frac{1}{2}\frac{\partial}{\partial x}\left(E_x^2 - E_y^2 - E_z^2\right) + \frac{\partial}{\partial y}(E_yE_x) + \frac{\partial}{\partial z}(E_zE_x)$$

$$= \frac{\partial}{\partial x}\left(E_x^2 - \frac{1}{2}E_x^2 - \frac{1}{2}E_y^2 - \frac{1}{2}E_z^2\right) + \frac{\partial}{\partial y}(E_yE_x) + \frac{\partial}{\partial z}(E_zE_x).$$

$$= \frac{\partial}{\partial x}\left(E_x^2 - \frac{1}{2}\vec{E}\cdot\vec{E}\right) + \frac{\partial}{\partial y}(E_yE_x) + \frac{\partial}{\partial z}(E_zE_x) \qquad \blacksquare$$

Thus we can define a vector

$$\vec{T}_x := \epsilon_0\left(E_xE_x - \frac{1}{2}\vec{E}\cdot\vec{E}\right)\hat{x} + \epsilon_0E_yE_x\,\hat{y} + \epsilon_0E_zE_x\,\hat{z}$$

We can then see that

$$\left(\vec{\nabla}\cdot\vec{E}\right)E_x = \vec{\nabla}\cdot\vec{T}_x$$

and

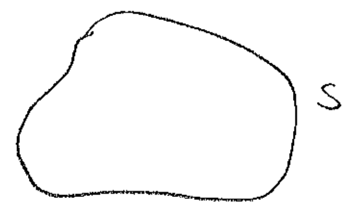
$$\frac{dP_{mech\,x}}{dt} = \int_{\text{region}} \vec{\nabla}\cdot\vec{T}_x\,d\tau = \oint_{\text{surface}} \vec{T}_x\cdot d\vec{a}$$

Thus we get:

Consider a closed region R with surface S

Suppose that there is no magnetic field present. Then given that the electric field is

$$\vec{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$$



define the electric stress vectors

$$\vec{T}_x = \epsilon_0 \left(E_x E_x - \tfrac{1}{2} \vec{E} \cdot \vec{E} \right) \hat{x} + \epsilon_0 E_y E_x \hat{y} + \epsilon_0 E_z E_x \hat{z}$$

$$\vec{T}_y = \epsilon_0 E_x E_y \hat{x} + \epsilon_0 \left(E_y E_y - \tfrac{1}{2} \vec{E} \cdot \vec{E} \right) \hat{x} + \epsilon_0 E_z E_y \hat{z}$$

$$\vec{T}_z = \epsilon_0 E_x E_z \hat{x} + \epsilon_0 E_y E_z \hat{y} + \epsilon_0 \left(E_z E_z - \tfrac{1}{2} \vec{E} \cdot \vec{E} \right) \hat{z}$$

Then we get that the total force on the region, or the rate of change of the mechanical momentum is:

$$\vec{F} = \frac{d\vec{p}_{mech}}{dt} = \left(\oint_{\text{surface}} \vec{T}_x \cdot d\vec{a} \right) \hat{x} + \left(\oint_S \vec{T}_y \cdot d\vec{a} \right) \hat{y} + \left(\oint_S \vec{T}_z \cdot d\vec{a} \right) \hat{z}$$

We can arrange the three vectors as follows:

$$\vec{T}_x \sim \epsilon_0 \begin{pmatrix} E_x E_x - \frac{1}{2} \vec{E} \cdot \vec{E} \\ E_x E_y \\ E_x E_z \end{pmatrix}$$

$$\vec{T}_y \sim \epsilon_0 \begin{pmatrix} E_y E_x \\ E_y E_y - \frac{1}{2} \vec{E} \cdot \vec{E} \\ E_y E_z \end{pmatrix} \qquad \text{etc...}$$

We can gather these into a matrix

$$\overleftrightarrow{T} = \epsilon_0 \begin{pmatrix} E_xE_x - \frac{1}{2}\vec{E}\cdot\vec{E} & E_xE_y & E_xE_z \\ E_yE_x & E_yE_y - \frac{1}{2}\vec{E}\cdot\vec{E} & E_yE_z \\ E_zE_x & E_zE_y & E_{zz} - \frac{1}{2}\vec{E}\cdot\vec{E} \end{pmatrix}$$

This is called the Maxwell electric stress tensor. We only need the stress tensor on the boundary of a region in order to determine the net force on all of the charge in that region.

Also note we use:

$$\overleftrightarrow{T} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

So, on a rectangular surface