Phys 312 2021

Lecture 8

Fri: HW 5pm

Tues 8.1 , 8.2.3 Super challenge!

Conservation laws in electromagnetism

As in any branch of physics, there are conservation laws in electromagnetism. These include:

1) conservation of charge
2) conservation of energy
3) conservation of momentum

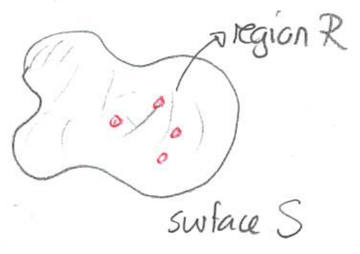
The latter two arise from Maxwell's equations plus notions of work and energy in classical mechanics. The first involves an additional assumption about charges and currents.

Charge conservation in electromagnetism

We assume that

1) all currents arise from moving charges
2) in a closed system net charge must remain constant.

Then consider a closed region. Let Q be the total charge in the region. Such charge may leave the region by flowing through the surface



This gives a net current through the surface

$$I = -\frac{\partial Q}{\partial t}$$

But if $\vec{J}$ is the current density on the region surface

ρ is " volume charge density inside the region

then

$$I = \oint_S \vec{J} \cdot d\vec{a}$$

$$Q = \int_R \rho(\vec{r},t)\, d\tau$$

$$\Rightarrow \quad \underbrace{\oint_S \vec{J} \cdot d\vec{a}}_{\int_R (\vec{\nabla} \cdot \vec{J})\, d\tau} = -\int_R \frac{\partial \rho}{\partial t}\, d\tau$$

$$\int_R (\vec{\nabla} \cdot \vec{J})\, d\tau = -\int_R \frac{\partial \rho}{\partial t}\, d\tau$$

This must be true for all regions. This implies that

$$\boxed{\text{Charge is conserved} \Leftrightarrow \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

This constrains $\vec{J}$ and ρ in relation to each other. So:

1) given a particular $\rho(\vec{r},t)$; $\vec{J}$ cannot be assigned arbitrarily

2) for a given $\rho(\vec{r},t)$ this does not completely fix $\vec{J}$.

Energy in electromagnetism

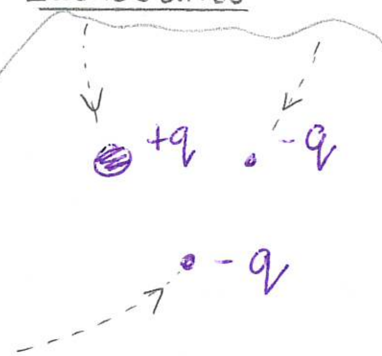
We have already seen that:

In electrostatics ⇝ work to assemble charge distribution
= energy stored in electric field

In magnetostatics ⇝ work to assemble charge distribution
= energy stored in magnetic field

We developed specific prescriptions for determining each of these. We aim to assemble them into a single energy quantity and demonstrate that they still apply in general time-dependent electromagnetism. To recap

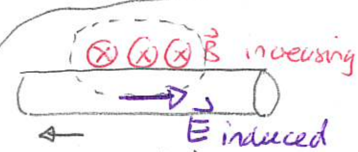
Electrostatics



assemble charges from infinitely far apart

$$W_{elec} = U_{elec} = \frac{\epsilon_0}{2}\int_{\text{all space}} \vec{E}\cdot\vec{E}\,d\tau$$

Magnetostatics.



current increases from zero

Assemble current against induced field

Increasing $\vec{B}$ field induces opposing $\vec{E}$ field.

Need to do work against field

$$W_{mag} = \frac{1}{2\mu_0}\int_{\text{all space}} \vec{B}\cdot\vec{B}\,d\tau$$

This suggests a total energy

$$\boxed{U = \frac{\epsilon_0}{2} \int_{\text{all space}} \vec{E} \cdot \vec{E} \, d\tau' + \frac{1}{2\mu_0} \int_{\text{all space}} \vec{B} \cdot \vec{B} \, d\tau'}$$

Sometimes we use

> The energy density in electric + magnetic fields is
>
> $$u = \frac{\epsilon_0}{2} \vec{E} \cdot \vec{E} + \frac{1}{2\mu_0} \vec{B} \cdot \vec{B}$$

Then the total energy is

$$\boxed{U = \int_{\text{all space.}} u \, d\tau'}$$

The process for extending this to all situations will be:

1) determine expressions for the mechanical work done by fields on charge distributions $\rightarrow W_{mech}$

2) Assume all charges are initially at rest. Then $K - K_i = W_{mech}$ $\Rightarrow K = W_{mech}$

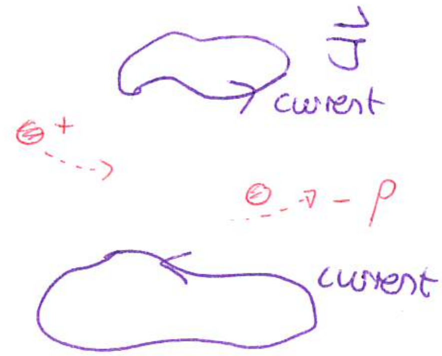
3) Show that $\frac{dU}{dt} + \frac{dW_{mech}}{dt} = 0$

$$\Rightarrow \frac{d}{dt}(U+K) = 0 \Rightarrow U + K = \text{constant}.$$

Mechanical work done by fields on charges

We consider a universe consisting of charges and currents that produce fields and exert forces on other charges + currents.

Then let $\vec{E}, \vec{B}$ be the fields produced by all such sources.



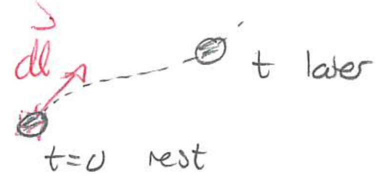
Now define

W_{mech} = work done by fields on distribution during some time interval $0 \rightarrow t$

In general

$$W_{mech} = \int \vec{F} \cdot \vec{dl}$$

trajectory from 0 to t



$q = \rho d\tau'$

$$= \int q \left[\vec{E} + \vec{v} \times \vec{B} \right] \cdot \vec{dl}$$

But $\vec{v}$ is parallel to $\vec{dl}$ so the magnetic field does no work.

Thus

$$W_{mech} = \int_{\text{path from } 0}^{t} q \, \vec{E} \cdot \vec{dl} = \int_0^t q \, \vec{E} \cdot \vec{v} \, dt$$

Then $q = \rho(\vec{r}, t) d\tau'$

$$\Rightarrow \quad W_{mech} = \int_0^t \int_{\text{all space}} \rho(\vec{r}, t) \vec{E} \cdot \vec{v} \, d\tau' \, dt$$

But $\rho \vec{v} = \vec{J}$ gives:

$$W_{mech} = \int_0^t \left\{ \int_{\text{all space}} \vec{E}\cdot\vec{J}\, d\tau' \right\} dt$$

We could differentiate w.r.t t and this gives:

$$\boxed{\frac{dW_{mech}}{dt} = \int_{\text{all space}} \vec{J}\cdot\vec{E}\, d\tau'}$$

How does this relate to kinetic energy? Let $K_0 =$ kinetic energy at time $t=0$ and $K(t) =$ kinetic energy at time t. Then

$$W_{mech} = K(t) - K_0$$

$$\Rightarrow \quad \frac{dW_{mech}}{dt} = \frac{dK}{dt}$$

$$\Rightarrow \quad \frac{dK}{dt} = \int_{\text{all space}} \vec{J}\cdot\vec{E}\, d\tau'$$

where K is the total kinetic energy of all matter

Note that $\frac{dW_{mech}}{dt} =$ power delivered

$$\Rightarrow \quad P = \int_{\text{all space}} \vec{J}\cdot\vec{E}\, d\tau' \quad \sim \text{ analogous to } P = VI$$

1 **Power to sustain a uniform current in a cylindrical wire**

A current with uniform density flows in the axial direction along a cylindrical wire with radius R. The current is sustained by an electric field which points in the direction of the current and is uniform.

a) Determine an expression for the potential difference between the ends of the wire in terms of the electric field and the length of the wire.

b) Using

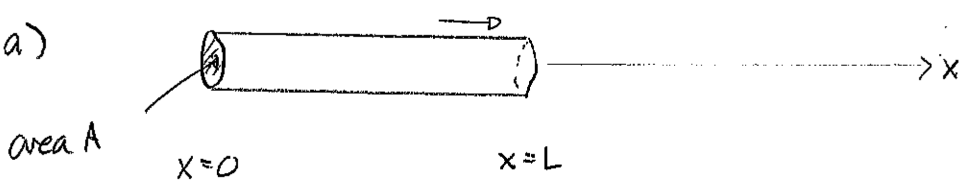
$$\frac{dW_{\text{mech}}}{dt} = \int \mathbf{E} \cdot \mathbf{J}\, d\tau'$$

for this situation, show that

$$P = I \Delta V$$

where I is the current that flows along the wire.

Answers: a)



We have $\Delta V = -\int \vec{E} \cdot d\vec{l}$

$$V(L) - V(0) = -\int \vec{E} \cdot d\vec{l}$$

Here $d\vec{l} = dx\, \hat{x}$ and $\vec{E} = E \hat{x}$ $\Rightarrow$ $V(L) - V(0) = -EL$

$\underset{\text{const}}{E}$

Let $\Delta V = V(0) - V(L)$. Thus $\Delta V = EL$

b) In this case $\vec{J} = \frac{I}{A} \hat{x} = \frac{I}{\pi R^2} \hat{x}$

Then $\int \vec{E} \cdot \vec{J}\, d\tau = \frac{I}{\pi R^2} E \int d\tau = \frac{I}{\pi R^2} E\, \pi R^2 L = EL\, I$

$$\Rightarrow \frac{dW_{mech}}{dt} = I \Delta V$$

$$\Rightarrow \boxed{P = I \Delta V}$$

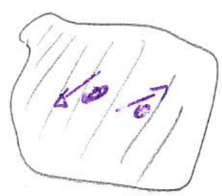
Energy Conservation

Generally energy conservation can take the forms:

1) all space - consider the mechanical work done on all charges
   - consider the energy stored in the fields in all space
   - energy is conserved if $\frac{dU}{dt} + \frac{dW_{mech}}{dt} = 0$

$$\Rightarrow \frac{dU}{dt} + \frac{dK}{dt} = 0$$

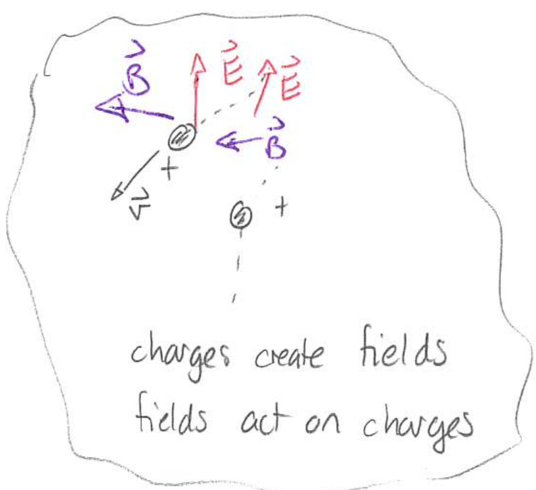
2) restricted region - consider mechanical work done on all charges in some region, W_{mech}



   - consider energy stored in some region U
   - then

$$\frac{dU}{dt} + \frac{dW_{mech}}{dt} = \text{rate of energy flow out of region}$$

Poynting's theorem will deal with both cases, and the second is more general. As a limited case we consider the first. In this situation



$\rightarrow$ Calculate total energy and rate of change of total energy in fields

$\hookrightarrow$ Compare to work done

2 Poynting's theorem: all space

The energy stored in the fields is

$$U = \frac{\epsilon_0}{2}\int \mathbf{E}\cdot\mathbf{E}\,\mathrm{d}\tau + \frac{1}{\mu_0 2}\int \mathbf{B}\cdot\mathbf{B}\,\mathrm{d}\tau.$$

Assume that the charges and currents that produce the fields are localized. Differentiate the energy with respect to time, substitute from Maxwell's equations and use vector calculus identities to show that

$$\frac{\partial U}{\partial t} = -\int \mathbf{E}\cdot\mathbf{J}\,\mathrm{d}\tau$$

provided that integration is done over all space.

Answer:

$$\frac{dU}{dt} = \frac{\epsilon_0}{2}\int \underbrace{\frac{\partial \vec{E}}{\partial t}}_{\text{same}}\cdot\vec{E}\,d\tau + \frac{\epsilon_0}{2}\int \vec{E}\cdot\underbrace{\frac{\partial \vec{E}}{\partial t}}\,d\tau$$

$$+ \frac{1}{2\mu_0}\int \underbrace{\frac{\partial \vec{B}}{\partial t}}_{\text{same}}\cdot\vec{B}\,d\tau + \frac{1}{2\mu_0}\int \vec{B}\cdot\underbrace{\frac{\partial \vec{B}}{\partial t}}\,d\tau$$

$$= \epsilon_0\int \frac{\partial \vec{E}}{\partial t}\cdot\vec{E}\,d\tau + \frac{1}{\mu_0}\int \frac{\partial \vec{B}}{\partial t}\cdot\vec{B}\,d\tau$$

Now $\vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad \dfrac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} \quad \Rightarrow \quad \epsilon_0\frac{\partial \vec{E}}{\partial t} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \vec{J}$$

So

$$\frac{dU}{dt} = \frac{1}{\mu_0}\int (\vec{\nabla}\times\vec{B})\cdot\vec{E}\,d\tau - \int \vec{J}\cdot\vec{E}\,d\tau - \frac{1}{\mu_0}\int (\vec{\nabla}\times\vec{E})\cdot\vec{B}\,d\tau$$

$$= \frac{1}{\mu_0}\int \left[(\vec{\nabla}\times\vec{B})\cdot\vec{E} - (\vec{\nabla}\times\vec{E})\cdot\vec{B}\right]d\tau - \int \vec{J}\cdot\vec{E}\,d\tau$$

Now a general vector identity is:

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

Thus: $\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$

$$\Rightarrow \qquad \frac{\partial u}{\partial t} = -\frac{1}{\mu_0}\int \vec{\nabla}\cdot(\vec{E}\times\vec{B})\,d\tau - \int \vec{J}\cdot\vec{E}\,d\tau$$

$$= -\frac{1}{\mu_0}\oint_{\text{surface}} (\vec{E}\times\vec{B})\cdot d\vec{a} - \int \vec{J}\cdot\vec{E}\,d\tau$$

We now assume that $\vec{E}\times\vec{B} \to \frac{1}{r^3}$ or faster as $r\to\infty$. Then the boundary term $\to 0$ at infinity. Thus

$$\frac{\partial u}{\partial t} = -\int \vec{J}\cdot\vec{E}\,d\tau$$

$$= -\frac{dW_{mech}}{dt} \qquad \boxtimes$$

To summarize:

Given that $\vec{E}, \vec{B}$ are produced by localized charge and current distributions then the total energy

$$U = \frac{\epsilon_0}{2} \int_{\text{all space}} \vec{E} \cdot \vec{E}\, d\tau + \frac{1}{2\mu_0} \int_{\text{all space}} \vec{B} \cdot \vec{B}\, d\tau$$

satisfies:

$$\frac{dU}{dt} + \frac{d W_{mech}}{dt} = 0$$

where the mechanical work done by the fields on the charge distribution satisfies

$$\frac{dW_{mech}}{dt} = \int \vec{J} \cdot \vec{E}\, d\tau$$

Then

$$\frac{dW_{mech}}{dt} = \frac{dK}{dt}$$

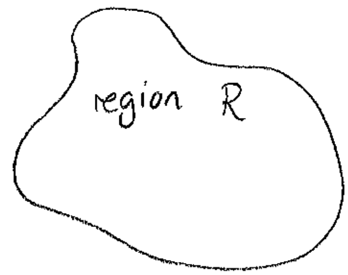
where K is the total kinetic energy of the matter implies:

$$\boxed{U + K = \text{constant}}$$

This is the conservation of energy.

Energy in a region of space

We can now extend the analysis to a finite region of space. We might expect that energy could enter or leave the region. Can we describe such energy flow?



The previous derivation gives:

$$\frac{dU_{region}}{dt} = -\frac{1}{\mu_0}\oint_{surface}(\vec{E}\times\vec{B})\cdot d\vec{a} - \int_{region}\vec{J}\cdot\vec{E}\,d\tau$$

Here

$$U_{region} = \frac{\epsilon_0}{2}\int_{region}\vec{E}\cdot\vec{E}\,d\tau + \frac{1}{2\mu_0}\int_{region}\vec{B}\cdot\vec{B}\,d\tau$$

is the energy stored in the fields in the region. Then we interpret the last integral as the rate of change of mechanical work and kinetic energy in the region. So

$$\frac{dU_{region}}{dt} + \frac{dK_{region}}{dt} = -\frac{1}{\mu_0}\oint_{surface}\vec{E}\times\vec{B}\cdot d\vec{a}$$

We define the Poynting vector as

$$\boxed{\vec{S} := \frac{1}{\mu_0}(\vec{E}\times\vec{B})}$$

This gives

$$\boxed{\frac{d}{dt}\left(U_{region} + K_{region}\right) = -\oint_{surface} \vec{S}\cdot d\vec{a}}$$

The Poynting vector will quantify the rate at which energy leaves or flows out of the region.