Lecture 4

Fri: HW by 5pm

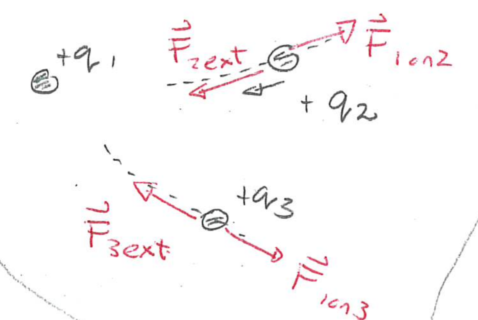
Tues: Read 6.1

HW Solutions

Energy and fields in matter

In general energy is required to assemble charge distributions. The basic notion involves:

charges initially at rest infinitely distant from each other → External forces are required to assemble charges eventually at rest



Then $W_{net} = \Delta K = 0$

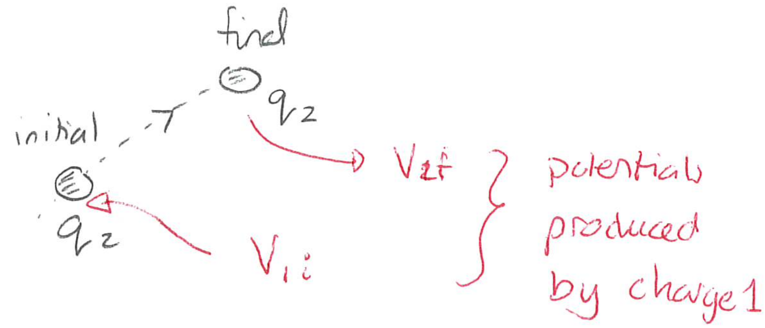
$$\Rightarrow W_{external} + W_{electrostatic} = 0 \quad \Rightarrow \quad W_{external} = -W_{electrostatic}.$$

Then we say

The energy stored in a charge distribution is the work done by external forces to assemble that distribution.

This can be calculated by calculating the work done by electrostatic forces

The basic rule that the work done by charge 1 on charge 2 is



$$W = -q_2 \Delta V_1$$

where ΔV_1 is the potential produced by charge 1.

This can be extended to a continuous distribution:

$$W = \frac{1}{2} \int_{\text{all space}} \rho(\vec{r}')\, V(\vec{r}')\, d\tau'$$

halves to remove double count.

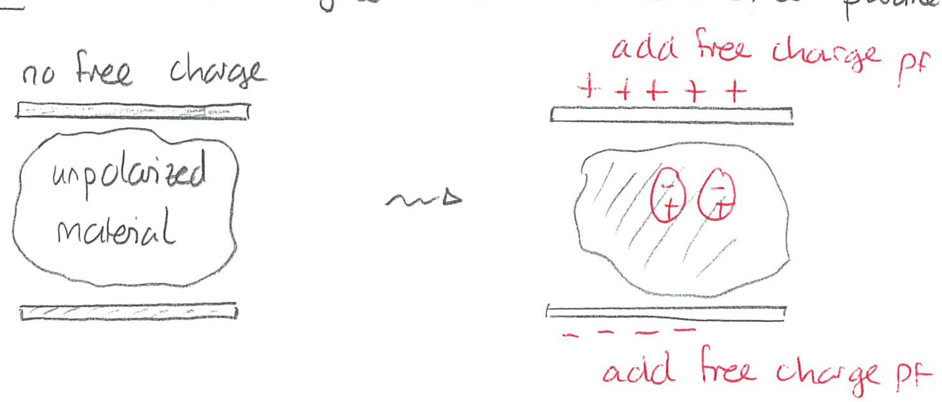
density of distribution

potential produced by distribution

Then using $\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}$ and integration by parts yields that:

> If a localized charge distribution produces electric field $\vec{E}$ then the work required to assemble this distribution is:
>
> $$W = \frac{\epsilon_0}{2} \int_{\text{all space}} \vec{E} \cdot \vec{E}\, d\tau$$

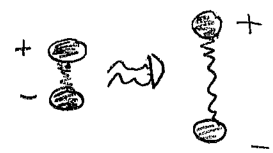
Now suppose that the free charges are in the presence of matter. We want to determine the work required to <u>assemble only the free charge</u>. We could imagine this in the case of a parallel plate capacitor



We might expect that for a dielectric the induced polarization will assist with the charge assembly and reduce the required energy. However, there will be several competing considerations

1) work done to assemble a new free charge against all existing charges, free and bound

2) work done to polarize molecules



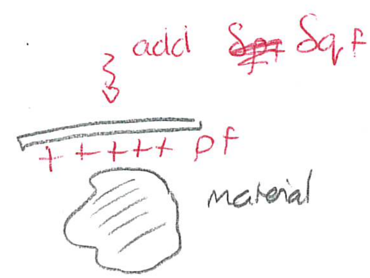
In general:

The work required to assemble free charge in the presence of any linear dielectric is:

$$W = \frac{1}{2} \int_{\text{all space}} \vec{D} \cdot \vec{E} \, d\tau$$

where $\vec{D}$ is the total electric displacement and $\vec{E}$ is the field produced by all charges

<u>Proof</u> Consider the process of adding free charge δq_f to an existing arrangement of free and bound charge.



The work required is

$$\delta W = \delta q_f \underbrace{V(\vec{r}\,')}_{\text{produced by existing charge}}$$

$$= \int \delta \rho_f \, d\tau' \, V(\vec{r}\,') = \int_{\text{all space.}} \delta \rho_f(\vec{r}\,') \, V(\vec{r}\,') \, d\tau'$$

Now

$$\rho_f = \vec{\nabla} \cdot \vec{D} \Rightarrow \delta \rho_f = \delta(\vec{\nabla} \cdot \vec{D}) = \vec{\nabla} \cdot \delta \vec{D}$$

where $\delta \vec{D}$ is the change in electric displacement just arising from charge $\delta \rho_f$. Thus

$$\delta W = \int_{\text{all space}} (\vec{\nabla} \cdot \delta \vec{D}) \, V(\vec{r}\,') \, d\tau'$$

Now consider integration by parts:

$$\vec{\nabla}\cdot(f\vec{A}) = (\vec{\nabla}f)\cdot\vec{A} + f\,\vec{\nabla}\cdot\vec{A} \Rightarrow f(\vec{\nabla}\cdot\vec{A}) = \vec{\nabla}\cdot(f\vec{A}) - (\vec{\nabla}f)\cdot\vec{A}$$

In this case $\vec{A} = \delta\vec{D}$

$f = V$

$$\Rightarrow \quad \delta W = \int_{\text{all space}} \vec{\nabla}\cdot(V\delta\vec{D})\,d\tau' - \int_{\text{all space}} (\vec{\nabla}V)\cdot\delta\vec{D}\,d\tau'$$

For localized charge distributes the first term integrates to zero. Then $\vec{\nabla}V = -\vec{E}$ implies:

$$\delta W = \int_{\text{all space}} (\delta\vec{D})\cdot\underbrace{\vec{E}}_{\text{produced by existing charge.}}\,d\tau'$$

Further simplification would require knowing how $\vec{D}$ and $\vec{E}$ are related. For a linear dielectric $\vec{D} = \frac{1}{\epsilon}\vec{E}$ and

$$(\delta\vec{D})\cdot\vec{E} = \frac{1}{\epsilon}(\delta\vec{E})\cdot\vec{E} = \frac{1}{\epsilon}\frac{1}{2}\,\delta(\vec{E}\cdot\vec{E}) = \frac{1}{2\cancel{\epsilon}}\,\delta(\cancel{\epsilon}\vec{D}\cdot\vec{E})$$

$$= \frac{1}{2}\,\delta(\vec{D}\cdot\vec{E})$$

Thus

$$\delta W = \frac{1}{2}\int_{\text{all space}} \delta(\vec{D}\cdot\vec{E})\,d\tau = \delta\left[\frac{1}{2}\int \vec{D}\cdot\vec{E}\,d\tau\right]$$

$$\Rightarrow \quad W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\,d\tau \qquad \boxtimes$$

1 Energy stored in a parallel plate capacitor with dielectric

A capacitor consists of two parallel plates, each with area A and separated by distance d, which is much smaller than the dimensions of either plate. The plates are charged uniformly with equal and opposite free charges of magnitude Q_f. The gap between the capacitors is filled with a linear dielectric with dielectric constant, ϵ.

a) Determine the energy stored in the capacitor in terms of the capacitance

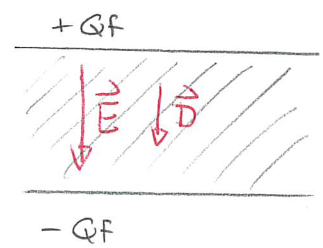
$$C = \frac{\epsilon A}{d}$$

and Q_f.

b) By what fraction is the energy different to that if the dielectric were absent?

Answer: We use

$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d\tau$$



which means we need $\vec{D}$, $\vec{E}$.

The field and electric displacement will only be non-zero between the plates. So we need these between the plates. In this region

$$\epsilon \vec{E} = \vec{D} \quad \Rightarrow \quad \vec{E} = \frac{1}{\epsilon}\vec{D}$$

and we can easily determine $\vec{D}$ from free charges. So

$$W = \frac{1}{2\epsilon}\int \vec{D}\cdot\vec{D}\, d\tau$$

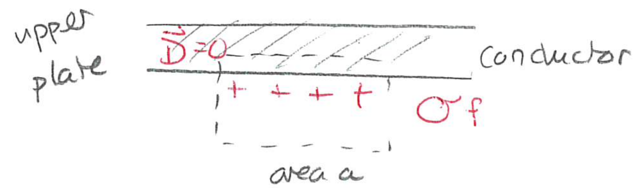
between plates.

We obtain $\vec{D}$ using

$$\oint_{\text{surface}} \vec{D}\cdot d\vec{a} = Q_{f\,enc}.$$

By symmetry

$$\vec{D} = D_z(z)\,\hat{z}$$



and we construct a pillbox around the upper plate as illustrated. Then

$$\oint \vec{D}\cdot d\vec{a} = q_{f\,enc} \Rightarrow -Da = q_{f\,enc}$$

$$\Rightarrow D = -\frac{q_{f\,enc}}{a} = -\frac{\sigma_{f\,enc}\,\cancel{a}}{\cancel{a}}$$

$$\Rightarrow \vec{D} = \begin{cases} -\sigma_{f\,enc}\,\hat{z} & \text{between plates} \\ 0 & \text{outside.} \end{cases}$$

Now $\sigma_{f\,enc} = \dfrac{Q_f}{A}$. Then:

$$\vec{D} = \begin{cases} -\dfrac{Q_f}{A}\,\hat{z} & \text{between plates} \\ 0 & \text{outside.} \end{cases}$$

Then

$$W = \frac{1}{2\epsilon}\int_{\text{between plates}} \vec{D}\cdot\vec{D}\,d\tau = \frac{Q_f^2}{2\epsilon A^2}\underbrace{\int_{\text{between plates}} d\tau'}_{Ad}$$

$$\Rightarrow W = \frac{1}{2}\frac{Q_f^2\,Ad}{\epsilon A^2} = \frac{1}{2}\frac{Q_f^2\,d}{\epsilon A} \Rightarrow \boxed{W = \frac{1}{2}\frac{Q_f^2}{C}}$$

b) Without dielectric $\epsilon \to \epsilon_0$

$$W_{\text{no dielectric}} = \frac{1}{2}\frac{1}{\epsilon_0}\frac{Q_f^2}{A}d$$

With

$$W_{\text{with dielectric}} = \frac{1}{2}\frac{1}{\epsilon}\frac{Q_f^2}{A}d$$

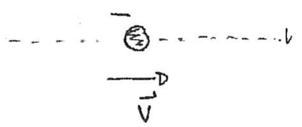
$$\Rightarrow \boxed{W_{\text{with dielectric}} = \frac{\epsilon_0}{\epsilon}W_{\text{without}}}$$

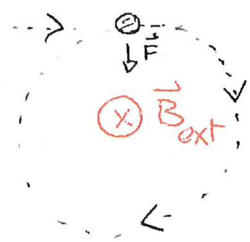
$$= \frac{\epsilon_0}{\epsilon_r\epsilon_0}W_{\text{without}}$$

$$\Rightarrow \boxed{W_{\text{with dielectric}} = \frac{1}{\epsilon_r}W_{\text{without}}}$$

Magnetism in Matter

Certain materials respond to magnetic fields by producing their own magnetic fields. Consider an electron (in a material) in the absence of and presence of an external magnetic field.





$\vec{F} = q\vec{v} \times \vec{B}$

no field ⇝ straight line

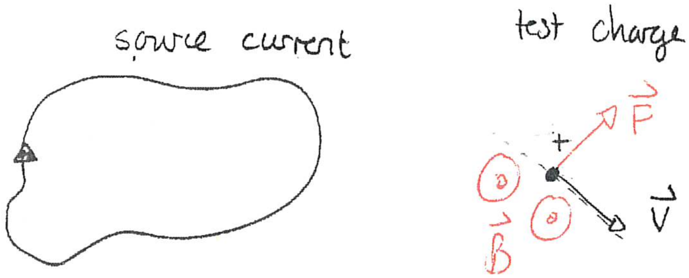
↳ like a current loop which produces a field $\vec{B}_{loop}$

The electron will produce a magnetic field opposite to that of the external field. This diamagnetic effect is commonplace and important in applications such as NMR.

We would like to consider these and other magnetic effects in materials. The basic model will be that the material consists of a collection of magnetic dipoles. We therefore review the magnetostatics associated with such dipoles

Magnetic fields + forces

Magnetostatics considers the forces exerted by stationary (time-independent) "source" currents on moving test charges



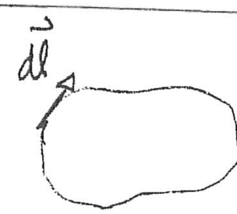
1) source current produces a magnetic field $\vec{B}$ at all locations

2) the magnetic field exerts a force on the test charge. The force is:

$$\boxed{\vec{F} = q\,(\vec{v} \times \vec{B})}$$

where q is the charge of the test charge
$\vec{v}$ " " velocity " " " "
$\vec{B}$ " " field produced by sources at the test charge

This can be generalized to forces on linear, surface and volume currents by integration:

linear current	$\vec{F} = \int_{\text{test current loop}} I\, d\vec{l} \times \vec{B}$	
surface current	$\vec{F} = \int \vec{K} \times \vec{B}\, da$	$\vec{K} = \sigma \vec{v}$ = surface current density
volume current	$\vec{F} = \int \vec{J} \times \vec{B}\, d\tau$	$\vec{J} = \rho \vec{v}$ = volume current density; ρ: charge density; $\vec{v}$: charge velocity

In general magnetic fields are produced by moving charges or currents. The rules by which these are determine are:

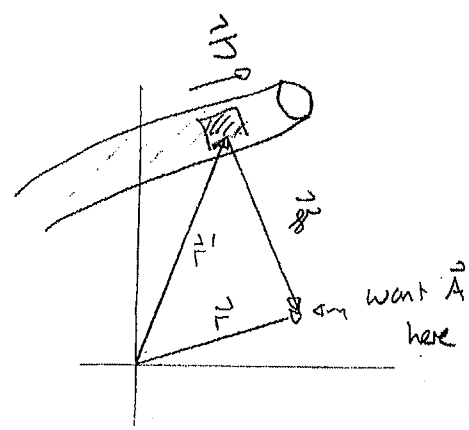
> If a source current does not vary with time then this produces a magnetic field $\vec{B}$ that satisfies
>
> $$\vec{\nabla}\cdot\vec{B} = 0$$
>
> $$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$
>
> where $\vec{J}$ is the current density vector.

There is also a potential formulation for magnetostatics. Specifically,

> Any current distribution produces a magnetic vector potential $\vec{A}$ such that
>
> $$\vec{B} = \vec{\nabla}\times\vec{A}$$

For currents that are restricted to a local region, the potential can be chosen so that it is generated as:

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}\,d\tau'$$



Magnetic Dipoles

The magnetic vector potential can be determined via a multipole expansion:

$$\vec{A}(\vec{r}) = \vec{A}_{dip}(\vec{r}) + \dots \quad \text{smaller terms}$$

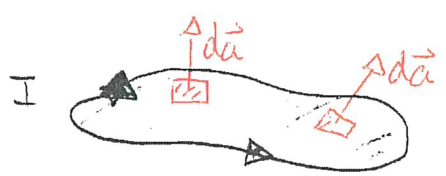
where the dipole term is:

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$$

and the magnetic dipole moment is for a (one dimensional) current loop is

$$\boxed{\vec{m} = I \int d\vec{a}}$$

Here $d\vec{a}$ is the integral over any surface that is enclosed by the loop.



Having calculated a dipole moment, one can compute the magnetic vector potential and eventually the resulting field using vector calculus

Eventually we obtain the result that:

The magnetic field produced by a magnetic dipole $\vec{m}$ at location $\vec{r}$ is:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^2}\left\{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right\}$$